

Feedback stabilization of the Cahn-Hilliard type system for phase separation

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Abstract. This article is concerned with the internal feedback stabilization of the phase field system of Cahn-Hilliard type, modeling the phase separation in a binary mixture. Under suitable assumptions on an arbitrarily fixed stationary solution, we construct via spectral separation arguments a feedback controller having its support in an arbitrary open subset of the space domain, such that the closed loop nonlinear system exponentially reach the prescribed stationary solution. This feedback controller has a finite dimensional structure in the state space of solutions. In particular, every constant stationary solution is admissible.

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1 Introduction

We consider the celebrated Cahn-Hilliard system (see [15], [20]), which is coupled, following the phase field approach introduced by Caginalp (see [10], [11]) with the energy balance equation in order to describe the spontaneous separation of the components in a binary mixture

$$(\theta + l_0\varphi)_t - \Delta\theta = 0, \text{ in } (0, \infty) \times \Omega, \quad (1.1)$$

$$\varphi_t - \Delta\mu = 0, \text{ in } (0, \infty) \times \Omega, \quad (1.2)$$

$$\mu = -\nu\Delta\varphi + F'(\varphi) - \gamma_0\theta, \text{ in } (0, \infty) \times \Omega. \quad (1.3)$$

We complete the system with standard homogeneous Neumann boundary conditions

$$\frac{\partial\theta}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = \frac{\partial\mu}{\partial\nu} = 0, \text{ on } (0, \infty) \times \partial\Omega \quad (1.4)$$

and with the initial data

$$\theta(0) = \theta_0, \varphi(0) = \varphi_0, \text{ in } \Omega. \quad (1.5)$$

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The equations and conditions (1.1)-(1.5) give rise to the so-called conserved phase field system, the name being due also to the mass conservation of φ , which is obtained by integrating (1.1) in space and time and using the boundary condition for μ in (1.4) and the initial condition for φ in (1.5). Proper references on conserved phase field system are [12], [13] by Caginalp, and the recent contributions [14] and [26], where a review of models and results is done as well. We also quote the contributions [17], [30] in which a conserved phase field model allowing further memory effects is investigated.

In the system (1.1)-(1.5) the variables θ , φ and μ represent the temperature, the order parameter and the chemical potential, respectively, ν is the outward normal vector to the boundary, l_0 , γ_0 are positive constants with some physical meaning, and F' is the derivative of the double-well potential

$$F(\varphi) = \frac{(\varphi^2 - 1)^2}{4}. \quad (1.6)$$

The space domain Ω is an open, bounded connected subset of \mathbb{R}^d , $d = 1, 2, 3$, with a sufficiently smooth boundary $\Gamma = \partial\Omega$, and the time t runs in $\mathbb{R}^+ = (0, \infty)$. This system has been widely studied in the last decades from several points of view including existence of attractors and optimal control. A list of recent references can be found in [16] and [18].

In this paper we shall treat the stabilization for the Cahn-Hilliard system around a stationary solution by two controllers (u, v) having their support in an open subset ω of Ω , and placed on the right-hand sides of equations (1.1)-(1.2). By introducing the expression of μ given by (1.3) into (1.2) the system to be stabilized reads

$$\varphi_t - \Delta(-\nu\Delta\varphi + F'(\varphi) - \gamma_0\theta) = 1_\omega^* v, \text{ in } (0, \infty) \times \Omega, \quad (1.7)$$

$$(\theta + l_0\varphi)_t - \Delta\theta = 1_\omega^* u, \text{ in } (0, \infty) \times \Omega, \quad (1.8)$$

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial(\Delta\varphi)}{\partial\nu} = \frac{\partial\theta}{\partial\nu} = 0, \text{ on } (0, \infty) \times \Gamma, \quad (1.9)$$

$$\varphi(0) = \varphi_0, \theta(0) = \theta_0, \text{ in } \Omega. \quad (1.10)$$

The second boundary condition in (1.9) follows by (1.3) and (1.4).

We specify that the function denoted 1_ω^* is chosen with the following properties

$$1_\omega^* \in C_0^\infty(\Omega), \quad \text{supp } 1_\omega^* \subset \omega, \quad 1_\omega^* > 0 \text{ on } \omega_0, \quad (1.11)$$

where ω_0 is an open subset of ω .

The purpose is to stabilize exponentially the solution to (1.7)-(1.10) around a stationary solution $(\varphi_\infty, \theta_\infty)$ of the uncontrolled system, by means of the feedback control (v, u) expressed as a function $\mathcal{F}(\varphi, \theta)$. This turns out to prove that

$$\lim_{t \rightarrow \infty} (\varphi(t), \theta(t)) = (\varphi_\infty, \theta_\infty), \quad (1.12)$$

with an exponential rate of convergence, provided that the initial datum (φ_0, θ_0) is in a suitable neighborhood of $(\varphi_\infty, \theta_\infty)$.

At this point we observe that the set of stationary states of the uncontrolled system (1.7)-(1.9) (for $u = v = 0$) is not empty, because this may have any constant solution θ_∞

with some constant or not constant solution φ_∞ . A discussion concerning the solutions to the stationary system

$$\begin{aligned} \nu \Delta^2 \varphi_\infty - \Delta F'(\varphi_\infty) &= 0, \text{ in } \Omega, \\ -\Delta \theta_\infty &= 0, \text{ in } \Omega, \\ \frac{\partial \varphi_\infty}{\partial \nu} &= \frac{\partial \Delta \varphi_\infty}{\partial \nu} = \frac{\partial \theta_\infty}{\partial \nu} = 0, \text{ on } \Gamma \end{aligned} \tag{1.13}$$

is presented in Lemma A1 in Appendix. The result asserts that θ_∞ is constant and $\varphi_\infty \in H^4(\Omega) \subset C^2(\overline{\Omega})$. Also, φ_∞ may be constant or not.

It should be mentioned that a simple analysis of the linearized system around a stationary state $(\varphi_\infty, \theta_\infty)$ reveals that, in general, not all solutions to the stationary system are asymptotically stable and so, their stabilization via a feedback controller with support in an arbitrary subset $\omega \subset \Omega$ is of crucial importance.

The stabilization technique used first in [31] for parabolic equations and then in [4], [5]-[7], [29] for Navier-Stokes equations and nonlinear parabolic systems is based on the design of the feedback controller as a linear combination of the unstable modes of the corresponding linearized system.

1.1 Main result

All the proofs given in this work converge to the main result of stabilization which is described below in a few words, for the reader's convenience. To this end, we briefly introduce some notation and definitions necessary to give the statement of the theorem. Some of them will be resumed and explained later, at the appropriate places.

Functional framework. Let us denote

$$H = L^2(\Omega), \quad V = H^1(\Omega),$$

with the standard scalar products, identify H with its dual space and set $V' = (H^1(\Omega))'$. Let $A : D(A) \subset H \rightarrow H$ be the linear operator

$$A = -\Delta + I, \quad D(A) = \left\{ w \in H^2(\Omega); \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma \right\}. \tag{1.14}$$

The operator A is m -accretive on H and so we can define its fractional powers A^α , $\alpha \geq 0$ (see e.g., [28], p. 72). We recall that A^α is a linear continuous positive and self-adjoint operator on H , with the domain

$$D(A^\alpha) = \{w \in H; \|A^\alpha w\|_H < \infty\}$$

and the norm

$$\|w\|_{D(A^\alpha)} = \|A^\alpha w\|_H. \tag{1.15}$$

Moreover, $D(A^\alpha) \subset H^{2\alpha}(\Omega)$, with equality if and only if $2\alpha < 3/2$.

Let F_l and γ be positive constants that will be specified later and let us denote by I the identity operator. We introduce the self-adjoint operator $\mathcal{A} : D(\mathcal{A}) \subset H \times H \rightarrow H \times H$,

$$\mathcal{A} = \begin{bmatrix} \nu \Delta^2 - F_l \Delta & \gamma \Delta \\ \gamma \Delta & -\Delta \end{bmatrix}, \tag{1.16}$$

having the domain

$$D(\mathcal{A}) = \left\{ w = (y, z) \in H^2(\Omega) \times H^1(\Omega); \mathcal{A}w \in H \times H, \frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma \right\}.$$

We denote by λ_i and $\{(\varphi_i, \psi_i)\}_{i \geq 1}$ the eigenvalues and eigenvectors respectively, of \mathcal{A} . Since \mathcal{A} is self-adjoint and its resolvent $(\lambda I + \mathcal{A})^{-1}$ is compact (as seen in a later proof), the eigenvalues are real and there is a finite number N of nonpositive eigenvalues $\lambda_i \leq 0$, $i = 1, \dots, N$. We introduce the operators B and B^* (B^* being the adjoint of B) as

$$B : \mathbb{R}^N \rightarrow H \times H, \quad B^* : H \times H \rightarrow \mathbb{R}^N,$$

$$BW = \begin{bmatrix} \sum_{i=1}^N 1_\omega^* \varphi_i w_i \\ \sum_{i=1}^N 1_\omega^* \psi_i w_i \end{bmatrix} \quad \text{for all } W = \begin{bmatrix} w_1 \\ \dots \\ w_N \end{bmatrix} \in \mathbb{R}^N, \quad (1.17)$$

and

$$B^*q = \begin{bmatrix} \int_\Omega 1_\omega^* (\varphi_1 q_1 + \psi_1 q_2) dx \\ \dots \\ \int_\Omega 1_\omega^* (\varphi_N q_1 + \psi_N q_2) dx \end{bmatrix} \quad \text{for all } q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in H \times H. \quad (1.18)$$

Moreover, let R be a linear positive self-adjoint operator

$$R : D(A^{1/2}) \times D(A^{1/4}) \rightarrow H \times H$$

which is the solution of the algebraic Riccati equation

$$2R\mathcal{A} + RBB^*R = \begin{bmatrix} A^3 & 0 \\ 0 & A^{3/2} \end{bmatrix}. \quad (1.19)$$

The existence of a solution R to (1.19) will be proved later on. Now, we are ready to present the stabilization result which is the main aim of our paper. We call the *closed loop system* the system (1.7)-(1.10) in which the right-hand side $(1_\omega^* v, 1_\omega^* u)$ is replaced by a function depending on (φ, θ) defined by the means of R , more exactly

$$(1_\omega^* v, 1_\omega^* u) = -BB^*R(\varphi - \varphi_\infty, \alpha_0(\theta - \theta_\infty + l(\varphi - \varphi_\infty))), \quad (1.20)$$

where the parameter α_0 is introduced below in (1.26). The theorem below given for the Cahn-Hilliard system in terms of (θ, φ) is a consequence of Theorem 3.1 in Section 3, which is the main result of this paper.

Let $(\theta_\infty, \varphi_\infty)$ be a solution to the stationary uncontrolled system (1.7)-(1.9) and set

$$\chi_\infty := \|\nabla \varphi_\infty\|_{L^\infty(\Omega)} + \|\Delta \varphi_\infty\|_{L^\infty(\Omega)}.$$

Theorem 1.1. *There exists $\chi_0 > 0$ (depending on the problem parameters, the domain and $\|\varphi_\infty\|_{L^\infty(\Omega)}$) such that the following holds true. If $\chi_\infty \leq \chi_0$, there exists ρ such that for all pairs $(\varphi_0, \theta_0) \in D(A^{1/2}) \times D(A^{1/4})$ with*

$$\|\varphi_0 - \varphi_\infty\|_{D(A^{1/2})} + \|\alpha_0(\theta_0 - \theta_\infty) + \alpha_0 l_0(\varphi_0 - \varphi_\infty)\|_{D(A^{1/4})} \leq \rho, \quad (1.21)$$

the closed loop system (1.7)-(1.10) with $(1_\omega^* v, 1_\omega^* u)$ replaced by (1.20) has a unique solution

$$\begin{aligned} (\varphi, \theta) \in & C([0, \infty); H \times H) \cap L^2(0, \infty; D(A^{3/2}) \times D(A^{3/4})) \\ & \cap W^{1,2}(0, \infty; (D(A^{1/2}) \times D(A^{1/4}))'), \end{aligned} \quad (1.22)$$

which is exponentially stable, that is

$$\begin{aligned} & \|\varphi(t) - \varphi_\infty\|_{D(A^{1/2})} + \|\alpha_0(\theta(t) - \theta_\infty) + \alpha_0 l_0(\varphi(t) - \varphi_\infty)\|_{D(A^{1/4})} \\ & \leq C_P e^{-kt} (\|\varphi_0\|_{D(A^{1/2})} + \|\theta_0\|_{D(A^{1/4})}), \end{aligned} \quad (1.23)$$

for some positive constants k and C_P .

In the previous relations the positive constants k and C_P depend on Ω , the problem parameters and $\|\varphi_\infty\|_{L^\infty(\Omega)}$. In addition, C_P depends on the full norm $\|\varphi_\infty\|_{W^{2,\infty}(\Omega)}$.

We remark that hypothesis $\chi_\infty \leq \chi_0$ is trivially satisfied if φ_∞ is a constant. Thus, any constant stationary solution can be stabilized. This is stressed in the following corollary.

Corollary 1.2. *Assume φ_∞ to be constant. Then, there exists ρ such that for all pairs $(\varphi_0, \theta_0) \in D(A^{1/2}) \times D(A^{1/4})$ satisfying (1.21) the unique solution to the closed loop system is exponentially stable.*

1.2 A few preliminaries and plan of the paper

We prefer to make a function transformation

$$\sigma = \alpha_0(\theta + l_0\varphi), \quad (1.24)$$

with $\alpha_0 > 0$ chosen such that

$$\frac{\gamma_0}{\alpha_0} = \alpha_0 l_0 =: \gamma > 0, \quad (1.25)$$

that is

$$\alpha_0 = \sqrt{\frac{\gamma_0}{l_0}}. \quad (1.26)$$

This transformation will give the possibility to work later on with a self-adjoint operator acting on the linear part of the system. We observe that if $l_0 = \gamma_0$ (which usually does not occur in the model) we directly obtain the self-adjoint linear operator.

Writing the system (1.7)-(1.9) in the variables φ and σ and using (1.25) and the notation

$$l := \gamma_0 l_0, \quad (1.27)$$

we get the equivalent nonlinear system

$$\varphi_t + \nu \Delta^2 \varphi - \Delta F'(\varphi) - l \Delta \varphi + \gamma \Delta \sigma = 1_\omega^* v, \text{ in } (0, \infty) \times \Omega, \quad (1.28)$$

$$\sigma_t - \Delta \sigma + \gamma \Delta \varphi = 1_\omega^* u, \text{ in } (0, \infty) \times \Omega, \quad (1.29)$$

$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial \Delta \varphi}{\partial \nu} = \frac{\partial \sigma}{\partial \nu} = 0, \text{ in } (0, \infty) \times \Gamma, \quad (1.30)$$

$$\varphi(0) = \varphi_0, \sigma(0) = \sigma_0 := \alpha_0(\theta_0 + l_0 \varphi_0), \text{ in } \Omega, \quad (1.31)$$

with the new meaning of u , namely, α_0 times the old u . We shall study in fact the stabilization for this transformed system. It is obvious that if the stabilization $\lim_{t \rightarrow \infty} (\varphi(t), \sigma(t)) = (\varphi_\infty, \sigma_\infty)$ is proved for system (1.28)-(1.31), whenever the initial datum (φ_0, σ_0) is in a neighborhood of $(\varphi_\infty, \sigma_\infty)$, then this implies the stabilization (1.12) for the corresponding system (1.7)-(1.10). We shall discuss this at the appropriate place. Here, σ_∞ is defined as $\alpha_0(\theta_\infty + l_0\varphi_\infty)$ and in general it can be constant or not, depending on the same property for φ_∞ . The stationary system in terms of φ_∞ and σ_∞ reads

$$\begin{aligned} \nu \Delta^2 \varphi_\infty - \Delta F'(\varphi_\infty) - l \Delta \varphi_\infty + \gamma \Delta \sigma_\infty &= 0, \text{ in } \Omega, \\ -\Delta \sigma_\infty + \gamma \Delta \varphi_\infty &= 0, \text{ in } \Omega, \\ \frac{\partial \varphi_\infty}{\partial \nu} &= \frac{\partial \Delta \varphi_\infty}{\partial \nu} = \frac{\partial \sigma_\infty}{\partial \nu} = 0, \text{ on } \Gamma. \end{aligned} \quad (1.32)$$

Next, we rewrite the difference between system (1.28)-(1.31) and system (1.32) by denoting

$$y = \varphi - \varphi_\infty, \quad z = \sigma - \sigma_\infty, \quad (1.33)$$

$$y_0 = \varphi_0 - \varphi_\infty, \quad z_0 = \sigma_0 - \sigma_\infty. \quad (1.34)$$

We have

$$y_t + \nu \Delta^2 y - \Delta(F'(y + \varphi_\infty) - F'(\varphi_\infty)) - l \Delta y + \gamma \Delta z = 1_\omega^* v, \text{ in } (0, \infty) \times \Omega, \quad (1.35)$$

$$z_t - \Delta z + \gamma \Delta y = 1_\omega^* u, \text{ in } (0, \infty) \times \Omega, \quad (1.36)$$

$$y(0) = y_0, \quad z(0) = z_0, \text{ in } \Omega, \quad (1.37)$$

$$\frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \text{ on } (0, \infty) \times \Gamma, \quad (1.38)$$

and we shall stabilize it around the state $(0, 0)$ for the initial datum (y_0, z_0) lying in a neighborhood of $(0, 0)$.

This result is formulated in Theorem 3.1. Since its proof is technical and long, we shall split parts of it in several Propositions, according to a strategy following the steps below. As we have already specified, a central part is the stabilization of the linearized system. We mention that actually this will be not exactly the linearized system corresponding to the nonlinear one, but the modified linear system (2.10)-(2.13) (given in Section 2), which is more convenient to be used in this case. Here are the steps:

(i) Proof of the stabilization of the linear system (2.10)-(2.13) by a finite dimensional control, in Proposition 2.2.

(ii) Introduction and representation of R , calculation of the feedback control involving the operators B , B^* and R , and stabilization of the linear system (2.10)-(2.13) by this feedback control in Section 2, Propositions 2.3, 2.4 and Remark 2.6.

(iii) Proof of the existence of a unique solution to the *nonlinear closed loop system* (1.35)-(1.38) (with (u, v) expressed in terms of (y, z) by means of B , B^* , R) and stabilization of this solution, in Section 3, Theorem 3.1. As a matter of fact, this is the main result of stabilization given for the system in (y, z) .

(iv) Retrieval of the result presented in Theorem 1.1 for the solution (φ, θ) , as a consequence of Theorem 3.1.

Notation. We denote by C or C_i , $i = 1, 2, \dots$ several positive constants possibly depending on the system structure (ν, l, γ) , domain, space dimension, and possibly on the norms of φ_∞ . However, we shall locally specify the dependence of the constants on φ_∞ . A symbol like C_δ with Greek subscripts denotes (possibly different) constants that depend on the respective parameter, in addition. Also, we mark precise constants which can be involved in essential proof arguments by certain (small or capital) letters and specify them in the text. Whenever no confusion may arise we shall not indicate the arguments of the functions, for simplicity.

We shall denote by (\cdot, \cdot) a pair in a product space and by $(\cdot, \cdot)_X$ the scalar product in a space X . The norms in $L^\infty(\Omega)$ and $W^{2,\infty}(\Omega)$ are indicated by $\|\cdot\|_\infty$ and $\|\cdot\|_{2,\infty}$, respectively.

Tools. We repeatedly use the Sobolev embedding theorems

$$\|w\|_{L^{2r}(\Omega)} \leq C \|w\|_{H^\alpha(\Omega)}, \quad \alpha \geq d \left(\frac{1}{2} - \frac{1}{2r} \right), \quad d \leq 3, \quad (1.39)$$

(see e.g., [9], p. 285), its consequence

$$\|w\|_\infty \leq C \|w\|_{H^2(\Omega)}, \quad (1.40)$$

and the elementary Young inequality

$$ab \leq \delta a^p + C_\delta b^{p'}, \quad a \geq 0, \quad b \geq 0, \quad \delta > 0, \quad p \in (1, \infty), \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad (1.41)$$

with C_δ depending on p , besides δ .

Moreover, we shall account for the following inequalities involving the powers of A :

$$\|A^\alpha w\|_H \leq C \|A^{\alpha_1} w\|_H^\lambda \|A^{\alpha_2} w\|_H^{1-\lambda}, \quad \text{for } \alpha = \lambda \alpha_1 + (1 - \lambda) \alpha_2, \quad \lambda \in [0, 1], \quad (1.42)$$

$$\|A^\alpha w\|_H \leq C \|A^\beta w\|_H, \quad \text{if } \alpha < \beta, \quad (1.43)$$

$$\|A^\alpha w\|_{H^\beta(\Omega)}^2 \leq C \|A^{\alpha+\beta/2} w\|_H^2, \quad (1.44)$$

with C depending on the domain and the exponents.

2 Stabilization of the linear system

In this section we shall deal with the linear system extracted from (1.35)-(1.38).

Let $\varphi_\infty \in C^2(\overline{\Omega})$ be the first component of a solution to the stationary problem (1.32).

We recall that F is defined in (1.6) and we develop $F'(y + \varphi_\infty)$ in Taylor expansion and rewrite (1.35) as

$$y_t + \nu \Delta^2 y - \Delta(F''(\varphi_\infty)y) - l \Delta y + \gamma \Delta z = \Delta F_r(y) + 1_\omega^* v, \quad (2.1)$$

where $F_r(y)$ is the rest of second order. Then, we define

$$\overline{F''}_\infty = \frac{1}{m_\Omega} \int_\Omega F''(\varphi_\infty(\xi)) d\xi = \frac{3}{m_\Omega} \int_\Omega \varphi_\infty^2(\xi) d\xi - 1, \quad (2.2)$$

where m_Ω is the measure of Ω . Thus, we have

$$F''(\varphi_\infty(x)) = \overline{F''}_\infty + g(x), \quad (2.3)$$

where

$$g(x) := \frac{1}{m_\Omega} \int_\Omega (F''(\varphi_\infty(x)) - F''(\varphi_\infty(\xi))) d\xi = \frac{3}{m_\Omega} \int_\Omega (\varphi_\infty^2(x) - \varphi_\infty^2(\xi)) d\xi. \quad (2.4)$$

Plugging (2.2) in (2.1) we get the following equivalent form of the nonlinear system (1.35)-(1.38)

$$y_t + \nu \Delta^2 y - F_l \Delta y + \gamma \Delta z = \Delta(F_r(y) + g(x)y) + 1_\omega^* v, \text{ in } (0, \infty) \times \Omega, \quad (2.5)$$

$$z_t - \Delta z + \gamma \Delta y = 1_\omega^* u, \text{ in } (0, \infty) \times \Omega, \quad (2.6)$$

$$y(0) = y_0, \quad z(0) = z_0, \text{ in } \Omega, \quad (2.7)$$

$$\frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \text{ in } (0, \infty) \times \Gamma, \quad (2.8)$$

where

$$F_l = \overline{F''}_\infty + l. \quad (2.9)$$

We note that F_l also depends on Ω and on $\|\varphi_\infty\|_{L^2(\Omega)}$.

Now, we introduce the linear system

$$y_t + \nu \Delta^2 y - F_l \Delta y + \gamma \Delta z = 1_\omega^* v, \text{ in } (0, \infty) \times \Omega, \quad (2.10)$$

$$z_t - \Delta z + \gamma \Delta y = 1_\omega^* u, \text{ in } (0, \infty) \times \Omega, \quad (2.11)$$

$$y(0) = y_0, \quad z(0) = z_0, \text{ in } \Omega, \quad (2.12)$$

$$\frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \text{ in } (0, \infty) \times \Gamma, \quad (2.13)$$

which is going to be studied in this Section, while the nonlinear system (2.5)-(2.8) will be the object of Section 3.

Recalling the definition (1.16) of the operator \mathcal{A} we can write (2.10)-(2.13) as

$$\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) = 1_\omega^* U(t), \text{ a.e. } t \in (0, \infty), \quad (2.14)$$

$$(y(0), z(0)) = (y_0, z_0), \quad (2.15)$$

where $U(t) = (v(t), u(t))$.

Since the domain Ω is regular enough it follows that $D(\mathcal{A}) \subset H^4(\Omega) \times H^2(\Omega)$. Also, we note that the operator \mathcal{A} is self-adjoint.

2.1 Stabilization of the linear system by a finite dimensional controller

We set

$$\mathcal{H} = H \times H, \quad \mathcal{V} = D(A) \times D(A^{1/2}), \quad \mathcal{V}' = (D(A) \times D(A^{1/2}))', \quad (2.16)$$

and note that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ algebraically and topologically, with compact injections. The scalar products on \mathcal{H} and \mathcal{V} are defined by

$$\begin{aligned} ((y, z), (\psi_1, \psi_2))_{\mathcal{H}} &= \int_{\Omega} (y\psi_1 + z\psi_2) dx, \\ ((y, z), (\psi_1, \psi_2))_{\mathcal{V}} &= (\Delta y, \Delta \psi_1)_H + (y, \psi_1)_H + (\nabla z, \nabla \psi_2)_{\mathcal{H}} + (z, \psi_2)_H. \end{aligned}$$

We note that the second scalar product is equivalent to the one induced on \mathcal{V} by the standard scalar product of $H^2(\Omega) \times H^1(\Omega)$.

Proposition 2.1. *The operator \mathcal{A} is quasi m -accretive on \mathcal{H} , that is $\lambda I + \mathcal{A}$ is m -accretive for some $\lambda > 0$, and its resolvent is compact.*

Let $(y_0, z_0) \in \mathcal{H}$ and $(v, u) \in L^2(0, T; \mathcal{H})$. Then, problem (2.14)-(2.15) has a unique solution

$$(y, z) \in C([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \cap W^{1,2}([0, T]; \mathcal{V}'), \text{ for all } T > 0. \quad (2.17)$$

Moreover, $(y, z) \in C((0, T]; \mathcal{V})$ and we have the estimate

$$\begin{aligned} &\|(y(t), z(t))\|_{\mathcal{H}}^2 + \|(y, z)\|_{L^2(0, T; \mathcal{V})}^2 + t \|(y(t), z(t))\|_{\mathcal{V}}^2 \\ &\leq C \left(\|(y_0, z_0)\|_{\mathcal{H}}^2 + \int_0^T \|1_{\omega}^* U(s)\|_{\mathcal{H}}^2 ds \right), \text{ for all } t \in (0, T]. \end{aligned} \quad (2.18)$$

The constant C depends on Ω , T , the problem parameters and $\|\varphi_{\infty}\|_{L^2(\Omega)}$.

Proof. The several constants we introduce in the proof possibly depend on $\|\varphi_{\infty}\|_{L^2(\Omega)}$. We still denote by \mathcal{A} the operator from \mathcal{V} to \mathcal{V}' given by

$$\begin{aligned} \langle \mathcal{A}(y, z), (\psi_1, \psi_2) \rangle_{\mathcal{V}', \mathcal{V}} &= \int_{\Omega} (\nu \Delta y \cdot \Delta \psi_1 + F_l \nabla y \cdot \nabla \psi_1 - \gamma \nabla z \cdot \nabla \psi_1) dx \\ &\quad + \int_{\Omega} (\nabla z \cdot \nabla \psi_2 - \gamma \nabla y \cdot \nabla \psi_2) dx, \end{aligned} \quad (2.19)$$

for any $(\psi_1, \psi_2) \in \mathcal{V}$. As a matter of fact this is the extension of \mathcal{A} defined by (1.16). We easily see that \mathcal{A} is bounded from \mathcal{V} to \mathcal{V}' , that is

$$\|\mathcal{A}(y, z)\|_{\mathcal{V}'} = \sup_{(\psi_1, \psi_2) \in \mathcal{V}, \|(\psi_1, \psi_2)\|_{\mathcal{V}} \leq 1} \left| \langle \mathcal{A}(y, z), (\psi_1, \psi_2) \rangle_{\mathcal{V}', \mathcal{V}} \right| \leq C \|(y, z)\|_{\mathcal{V}}, \quad (2.20)$$

and that

$$\langle \mathcal{A}(y, z), (y, z) \rangle_{\mathcal{V}', \mathcal{V}} \geq C_1 \|(y, z)\|_{\mathcal{V}}^2 - C_2 \|(y, z)\|_{\mathcal{H}}^2, \text{ for all } (y, z) \in \mathcal{V}, \quad (2.21)$$

because

$$\begin{aligned}
\langle \mathcal{A}(y, z), (y, z) \rangle_{\mathcal{V}', \mathcal{V}} &= \int_{\Omega} (\nu |\Delta y|^2 + F_l |\nabla y|^2 - 2\gamma \nabla y \cdot \nabla z + |\nabla z|^2) dx \\
&\geq \nu \|\Delta y\|_H^2 - (|F_l| + 2\gamma^2) \|\nabla y\|_H^2 + \frac{1}{2} \|\nabla z\|_H^2 \\
&= \nu \|y\|_{D(A)}^2 + \frac{1}{2} \|z\|_{H^1(\Omega)}^2 - \nu \|y\|_H^2 - \frac{1}{2} \|z\|_H^2 - a_0 \|\nabla y\|_H^2,
\end{aligned}$$

with $a_0 = |F_l| + 2\gamma^2$. Since

$$a_0 \|\nabla y\|_H^2 \leq C \|y\|_{D(A)} \|y\|_H \leq \frac{\nu}{2} \|y\|_{D(A)}^2 + \frac{C^2}{2\nu} \|y\|_H^2$$

we obtain (2.21). Based on these properties, the restriction of \mathcal{A} to \mathcal{H} , $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, previously defined, is quasi m -accretive on \mathcal{H} . This means that $\mathcal{A} + C_2 I : \mathcal{V} \rightarrow \mathcal{V}'$ is coercive, thus surjective (see [3], p. 36).

Let $(y_0, z_0) \in \mathcal{H}$ and set $Y = (y, z)$. Since $1_\omega^* U \in L^2(0, T; \mathcal{H})$ and \mathcal{A} is symmetric, by the Lions existence theorem (see [23], Thm. 1.1, p. 46) problem (2.14)-(2.15) has a unique solution satisfying (2.17) and

$$\|Y(t)\|_{\mathcal{H}}^2 + \int_0^t \|Y(s)\|_{\mathcal{V}}^2 ds \leq C \left(\|Y_0\|_{\mathcal{H}}^2 + \int_0^t \|1_\omega^* U(s)\|_{\mathcal{H}}^2 ds \right), \quad \forall t \in [0, T]. \quad (2.22)$$

This is a part of (2.18) and the constant C also depends on T . Now, let us multiply formally (2.14) by $t \frac{dY}{dt}(t)$ scalarly in \mathcal{H} . Since \mathcal{A} is symmetric we have

$$t \left\| \frac{dY}{dt}(t) \right\|_{\mathcal{H}}^2 + \frac{1}{2} \frac{d}{dt} t \langle \mathcal{A}Y(t), Y(t) \rangle_{\mathcal{V}', \mathcal{V}} = \frac{1}{2} \langle \mathcal{A}Y(t), Y(t) \rangle_{\mathcal{V}', \mathcal{V}} + \left(1_\omega^* U(t), t \frac{dY}{dt}(t) \right)_{\mathcal{H}}.$$

By integrating in time and applying the Young inequality we easily get

$$\int_0^t s \left\| \frac{dY}{ds}(s) \right\|_{\mathcal{H}}^2 ds + t \langle \mathcal{A}Y(t), Y(t) \rangle_{\mathcal{V}', \mathcal{V}} \leq \int_0^t \langle \mathcal{A}Y(s), Y(s) \rangle_{\mathcal{V}', \mathcal{V}} ds + \int_0^t \|1_\omega^* U(s)\|_{\mathcal{H}}^2 ds.$$

By (2.21) and (2.20) we obtain

$$C_1 t \|Y(t)\|_{\mathcal{V}}^2 - C_2 t \|Y(t)\|_{\mathcal{H}}^2 \leq C \int_0^t \|Y(s)\|_{\mathcal{V}}^2 ds + \int_0^t \|1_\omega^* U(s)\|_{\mathcal{H}}^2 ds,$$

whence, by (2.22) we finally get the complete estimate (2.18) as claimed.

The above argument shows that $(\lambda I + \mathcal{A})^{-1}$ is well defined for $\lambda \geq C_2$.

Let $(f_1, f_2) \in \mathcal{H}$ and denote $(\lambda I + \mathcal{A})^{-1}(f_1, f_2) = (y, z)$. It is readily seen that (2.21) implies

$$\|(y, z)\|_{\mathcal{V}}^2 \leq C \|(f_1, f_2)\|_{\mathcal{H}}^2, \quad \text{for } \lambda \geq C_2,$$

and some $C > 0$, whence it follows that $(\lambda I + \mathcal{A})^{-1}(E)$ is relatively compact whenever E is bounded in \mathcal{H} . \square

We recall now that λ_i and $\{(\varphi_i, \psi_i)\}_{i \geq 1}$ are the eigenvalues and eigenvectors of \mathcal{A} ,

$$\mathcal{A}(\varphi_i, \psi_i) = \lambda_i(\varphi_i, \psi_i), \quad i = 1, 2, \dots \quad (2.23)$$

We notice that one of the coefficients of \mathcal{A} (see (2.19)) is F_l . Hence the eigenvalues and the eigenfunctions depend also on $\|\varphi_\infty\|_{L^2(\Omega)}$.

Since \mathcal{A} is self-adjoint, its eigenvalues are real and semi-simple, that is, \mathcal{A} is diagonalizable (see [21], p. 59). The eigenvectors corresponding to distinct eigenvalues are orthogonal. Then, orthogonalizing the system $\{(\varphi_i, \psi_i)\}_i$ in the space \mathcal{H} we may assume that it is orthonormal and complete in \mathcal{H} . Moreover, since the resolvent of \mathcal{A} is compact, there exists a finite number of nonpositive eigenvalues (see [21], p. 187). The sequence $\{\lambda_i\}_i$ is increasing and every eigenvalue is repeated according to its order of multiplicity. Let N be the number of these nonpositive eigenvalues, that is $\lambda_i \leq 0$, for $i = 1, \dots, N$.

Next we show that the system (2.14)-(2.15) can be stabilized by a finite dimensional control U of the form

$$U(t, x) = (v(t, x), u(t, x)) = \sum_{j=1}^N w_j(t)(\varphi_j(x), \psi_j(x)). \quad (2.24)$$

We rewrite (2.14)-(2.15) as an *open loop linear system*

$$\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) = \sum_{j=1}^N w_j(t)1_\omega^*(\varphi_j, \psi_j), \quad \text{a.e. } t \in (0, \infty), \quad (2.25)$$

and take an arbitrary initial condition in \mathcal{H} ,

$$(y(0), z(0)) = (y^0, z^0). \quad (2.26)$$

Proposition 2.2. *There exist $w_j \in L^2(\mathbb{R}^+)$, $j = 1, \dots, N$, such that the controller*

$$U(t, x) = (v(t, x), u(t, x)) = \sum_{j=1}^N w_j(t)(\varphi_j(x), \psi_j(x)), \quad t \geq 0, \quad x \in \Omega, \quad (2.27)$$

stabilizes exponentially system (2.25)-(2.26), that is, its solution (y, z) satisfies

$$\|y(t)\|_H + \|z(t)\|_H \leq C_P e^{-kt} (\|y^0\|_H + \|z^0\|_H), \quad \text{for all } t \geq 0. \quad (2.28)$$

Moreover, we have

$$\left(\sum_{j=1}^N \int_0^\infty |w_j(t)|^2 dt \right)^{1/2} \leq C_P (\|y^0\|_H + \|z^0\|_H). \quad (2.29)$$

In both formulas C_P and k depend on the problem parameters (ν, γ, l, Ω) and $\|\varphi_\infty\|_{L^2(\Omega)}$.

Proof. Let $T_0 > 0$ be arbitrary but fixed. We prove that the solution is represented by a sum of two pairs of functions such that the functions in the first pair vanish at $t = T_0$ and the functions in the second pair decrease exponentially to 0, as $t \rightarrow \infty$. We split the proof in two parts.

Part 1. We have the representation

$$(y(t, x), z(t, x)) = \sum_{j=1}^{\infty} \xi_j(t) (\varphi_j(x), \psi_j(x)), \quad (t, x) \in (0, \infty) \times \Omega, \quad (2.30)$$

with $\xi_j \in C(\mathbb{R}^+)$ and

$$\xi_i(0) = \xi_{i0} := \int_{\Omega} (y^0 \varphi_j(x) + z^0 \psi_j(x)) dx, \quad i \geq 1. \quad (2.31)$$

We plug the expressions (2.30) into (2.25)-(2.26), getting

$$\sum_{j=1}^{\infty} (\xi_j'(t) (\varphi_j, \psi_j) + \xi_j(t) \lambda_j (\varphi_j, \psi_j)) = \sum_{j=1}^N 1_{\omega}^* w_j(t) (\varphi_j, \psi_j).$$

Taking into account that $((\varphi_i, \psi_i), (\varphi_j, \psi_j))_{H \times H} = \delta_{ij}$, and multiplying scalarly the previous equation by (φ_i, ψ_i) in $H \times H$, we obtain

$$\xi_i' + \lambda_i \xi_i = \sum_{j=1}^N w_j d_{ij}, \quad \xi_i(0) = \xi_{i0}, \quad \text{for } i \geq 1, \quad (2.32)$$

with

$$d_{ij} = \int_{\Omega} 1_{\omega}^* (\varphi_i \varphi_j + \psi_i \psi_j) dx, \quad j = 1, \dots, N, \quad i = 1, \dots \quad (2.33)$$

Notice that $|d_{ij}| \leq \sup |1_{\omega}^*|$. First, we discuss the subsystem extracted from (2.32) by taking $i = 1, \dots, N$. It can be written in the form

$$X' + MX = DW \quad \text{and} \quad X(0) = X_0, \quad (2.34)$$

where

$$M = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_N \end{bmatrix}, \quad X = \begin{bmatrix} \xi_1 \\ \dots \\ \xi_N \end{bmatrix}, \quad X_0 = \begin{bmatrix} \xi_{10} \\ \dots \\ \xi_{N0} \end{bmatrix},$$

$$D = \begin{bmatrix} d_{11} & \dots & d_{1N} \\ \dots & \dots & \dots \\ d_{N1} & \dots & d_{NN} \end{bmatrix}, \quad W = \begin{bmatrix} w_1 \\ \dots \\ w_N \end{bmatrix}.$$

In the matrix M each λ_j is repeated according to its order of multiplicity.

Next, we prove that, for every $T_0 > 0$, system (2.32) for $i = 1, \dots, N$, is null controllable on $[0, T_0]$. To do that, we first show that the system $\{\sqrt{1_{\omega}^*} \varphi_j, \sqrt{1_{\omega}^*} \psi_j\}_{j=1}^N$ is linearly independent in ω (since $\text{supp } 1_{\omega}^* \subset \omega$). We assume that $\sum_{j=1}^N \alpha_j (\sqrt{1_{\omega}^*} \varphi_j, \sqrt{1_{\omega}^*} \psi_j) = 0$ in ω and deduce that

$\alpha_j = 0$ for $j = 1, \dots, N$. Our assumption reads $\sqrt{1_{\omega}^*} S = 0$ in ω where $S := \sum_{j=1}^N \alpha_j (\varphi_j, \psi_j)$.

Thus, $S = 0$ in the open set ω_0 since here $1_{\omega}^* > 0$. Now, we observe that the elliptic system

$$\nu \Delta^2 \varphi - F_l \Delta \varphi + \gamma \Delta \psi - \lambda_j \varphi = \gamma \Delta \varphi - \Delta \psi - \lambda_j \psi = 0$$

has constant (thus analytic) coefficients, and so any solution (φ, ψ) to it is analytic (see [27]). Thus, S is analytic too, whence $S = 0$ in Ω . This implies that $\alpha_j = 0$ for $j = 1, \dots, N$, since the system $\{(\varphi_j, \psi_j)\}_{j=1}^N$ is linearly independent in Ω .

In conclusion, the system $\{(\sqrt{I_\omega^*} \varphi_i, \sqrt{I_\omega^*} \psi_i)\}_i$ is linearly independent on ω and so, the determinant of $[d_{ij}]_{i,j}$ is not zero. This implies that any solution to

$$\sum_{i=1}^N d_{ij} p_i(t) = 0, \quad t \in [0, T_0], \quad j = 1, \dots, N, \quad (2.35)$$

must be zero, that is $p_i(t) = 0$ for all $i = 1, \dots, N$. So, the assumptions of Lemma A2 in Appendix are trivially satisfied, whence it follows that there are w_i such that $\xi_i(T_0) = 0$ for all $i = 1, \dots, N$, and

$$\left(\int_0^{T_0} \sum_{i=1}^N |w_i(t)|^2 dt \right)^{1/2} \leq C \sum_{i=1}^N |\xi_{i0}|, \quad (2.36)$$

where ξ_i , $i = 1, \dots, N$, denote the solution to system (2.32). It follows by (2.30) that $(y(T_0), z(T_0)) = (0, 0)$. By (2.27) and (2.31) we have

$$\begin{aligned} \left(\int_0^{T_0} (\|v(t)\|_H^2 + \|u(t)\|_H^2) dt \right)^{1/2} &= \left(\int_0^{T_0} \sum_{i=1}^N |w_i(t)|^2 dt \right)^{1/2} \\ &\leq C \sum_{i=1}^N |\xi_{i0}| \leq C (\|y^0\|_H + \|z^0\|_H). \end{aligned} \quad (2.37)$$

From (2.32), by the formula of variation of constants, we have

$$\xi_i(t) = e^{-\lambda_i t} \xi_{i0} + \sum_{j=1}^N d_{ij} \int_0^t e^{-\lambda_i(t-s)} w_j(s) ds, \quad t \geq 0$$

and recalling (2.36), we easily deduce the estimate

$$|\xi_i(t)| \leq C (\|y^0\|_H + \|z^0\|_H), \quad \text{for } t \in [0, T_0] \text{ and } i = 1, \dots, N. \quad (2.38)$$

The finite dimensional controller steers into the origin, at $t = T_0$, the solution $\{\xi_j\}_{j=1}^N$. We extend w_i and ξ_i by 0 at the right of $t = T_0$, and take as a new controller

$$\tilde{U}(t) = \begin{cases} (v(t), u(t)) & \text{for } t < T_0 \\ 0, & \text{for } t \geq T_0 \end{cases} \quad (2.39)$$

and $(y(t), z(t)) = (0, 0)$ for $t \geq T_0$. For this controller, (2.37) remains valid if we replace T_0 by $+\infty$. What we have obtained is exactly (2.29).

Part 2. We come back to (2.32) and discuss it for $i \geq N+1$. We show that it is stabilized exponentially in origin by the finite dimensional controller (2.39). Now, we assume $t > T_0$ and recall that $w_j(t) = 0$ for $t > T_0$. In (2.32) we apply again by the formula of variation of

constants and compute an estimate for ξ_i . Taking into account the fact that λ_i is positive for $i \geq N+1, \dots$, and $\lambda_{N+1} \leq \lambda_i$ for $i \geq N+2$, we have

$$\begin{aligned} |\xi_i(t)| &\leq e^{-\lambda_{N+1}t} |\xi_{i0}| + CN \int_0^{T_0} e^{-\lambda_{N+1}(t-s)} |w_j(s)| ds \\ &\leq e^{-\lambda_{N+1}t} |\xi_{i0}| + C_1 e^{-\lambda_{N+1}t} \left(\int_0^{T_0} e^{2\lambda_{N+1}s} ds \right)^{1/2} \left(\int_0^{T_0} |w_j(s)|^2 ds \right)^{1/2} \\ &\leq C_2 e^{-\lambda_{N+1}t} \left(|\xi_{i0}| + \left(\int_0^{T_0} |w_j(s)|^2 ds \right)^{1/2} \right), \text{ for } t > T_0, i \geq N+1, \end{aligned}$$

where C, C_1, C_2 are constants independent of (y^0, z^0) . Hence, by (2.30), (2.36) and by the Bessel inequality we obtain

$$\|y(t)\|_H + \|z(t)\|_H \leq C (\|y^0\|_H + \|z^0\|_H), \text{ for } t > T_0.$$

For $t \leq T_0$ we have a similar estimate as in (2.38) and all these together lead to (2.28), as claimed. \square

2.2 Feedback stabilization of the linear system

This subsection is devoted to the determination of a feedback controller (depending on the solution (y, z)) which stabilizes exponentially the solution to (2.25)-(2.26). We begin with the study of a minimization problem which is the key for this purpose.

We recall the operator A defined by (1.14) and consider the quadratic minimization problem

$$\Phi(y^0, z^0) = \min_{W \in L^2(0, \infty; \mathbb{R}^N)} \left\{ J(W) = \frac{1}{2} \int_0^\infty \left(\|A^{3/2}y(t)\|_H^2 + \|A^{3/4}z(t)\|_H^2 + \|W(t)\|_{\mathbb{R}^N}^2 \right) dt \right\} \quad (2.40)$$

subject to (2.25)-(2.26). Here W is the function $(w_1, \dots, w_N) \in L^2(0, \infty; \mathbb{R}^N)$ occurring in (2.25). We note that $D(\Phi) = \{(y^0, z^0) \in H \times H; \Phi(y^0, z^0) < \infty\}$.

In the next proofs we may also refer to $U = (v, u) \in L^2(0, \infty; H \times H)$ given by (2.27), where $\{(\varphi_j, \psi_j)\}_{j=1}^N$ are the eigenvectors of the operator \mathcal{A} corresponding to the unstable eigenvalues $\lambda_j \leq 0$.

The constants we shall introduce can depend on $\|\varphi_\infty\|_{L^2(\Omega)}$.

Proposition 2.3. *For each pair $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$, problem (2.40) has a unique optimal solution*

$$(\{w_j^*\}_{j=1}^N, y^*, z^*) \in L^2(\mathbb{R}^+; \mathbb{R}^N) \times L^2(\mathbb{R}^+; D(A^{3/2})) \times L^2(\mathbb{R}^+; D(A^{3/4})) \quad (2.41)$$

and

$$c_1 \left(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2 \right) \leq \Phi(y^0, z^0) \leq c_2 \left(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2 \right). \quad (2.42)$$

If $(y^0, z^0) \in D(A) \times D(A^{1/2})$, we have

$$\begin{aligned} & \left(\|Ay^*(t)\|_H^2 + \|A^{1/2}z^*(t)\|_H^2 \right) + \int_0^t \left(\|A^2y^*(s)\|_H^2 + \|Az^*(s)\|_H^2 \right) ds \\ & \leq c_3 \left(\|Ay^0\|_H^2 + \|A^{1/2}z^0\|_H^2 \right), \text{ for all } t \geq 0, \end{aligned} \quad (2.43)$$

where c_1, c_2, c_3 are positive constants (depending on Ω , the problem parameters and the quantity $\|\varphi_\infty\|_{L^2(\Omega)}$).

Proof. For all $(y^0, z^0) \in H \times H$, it follows by Proposition 2.2 that there exist $w_j \in L^2(\mathbb{R}^+)$ such that (2.25)-(2.26) has a solution with the properties (2.28)-(2.29).

First, we rewrite (2.25) by calculating the operator \mathcal{A} given by (1.16) in terms of the operator $A = -\Delta + I$. We get

$$\mathcal{A}(y, z) = \begin{bmatrix} \nu A^2 y + (F_l - 2\nu)Ay - (F_l - \nu)y - \gamma Az + \gamma z \\ -\gamma Ay + \gamma y + Az - z \end{bmatrix}. \quad (2.44)$$

Also, we recall the interpolation relations (1.42)-(1.43).

Now, let $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$. We multiply (2.25), where for the moment the right-hand side is written for simplicity $(1_\omega^* v, 1_\omega^* u)$, by $(Ay(t), A^{1/2}z(t))$ scalarly in $H \times H$ and obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|A^{1/2}y(t)\|_H^2 + \|A^{1/4}z(t)\|_H^2 \right) + \nu \|A^{3/2}y(t)\|_H^2 + \|A^{3/4}z(t)\|_H^2 \\ & = -(F_l - 2\nu)(Ay(t), Ay(t))_H + (F_l - \nu)(y(t), Ay(t))_H + \gamma(Az(t), Ay(t))_H \\ & \quad - \gamma(z(t), Ay(t))_H + \gamma(Ay(t), A^{1/2}z(t))_H - \gamma(y(t), A^{1/2}z(t))_H + (z(t), A^{1/2}z(t))_H \\ & \quad + \int_\Omega 1_\omega^* v(t) Ay(t) dx + \int_\Omega 1_\omega^* u(t) A^{1/2}z(t) dx, \text{ a.e. } t > 0. \end{aligned} \quad (2.45)$$

Next, we use the interpolation properties (1.42), (1.43) and the Young formula for the following terms:

$$\begin{aligned} -(F_l - 2\nu)(Ay(t), Ay(t))_H & \leq |F_l - 2\nu| \|Ay(t)\|_H^2 \leq C \left(\|A^{3/2}y(t)\|_H^{2/3} \|A^0y(t)\|_H^{1/3} \right)^2 \\ & \leq \delta \|A^{3/2}y(t)\|_H^2 + C_\delta \|y(t)\|_H^2, \end{aligned}$$

$$\begin{aligned} (F_l - \nu)(y(t), Ay(t))_H & \leq |F_l - \nu| \|y(t)\|_H \|Ay(t)\|_H \\ & \leq C \|y(t)\|_H \left(\|A^{3/2}y(t)\|_H^{2/3} \|y(t)\|_H^{1/3} \right) \\ & = C \|A^{3/2}y(t)\|_H^{2/3} \|y(t)\|_H^{4/3} \leq \delta \|A^{3/2}y(t)\|_H^2 + C_\delta \|y(t)\|_H^2, \end{aligned}$$

$$\begin{aligned} \gamma(Az(t), Ay(t))_H & = \gamma(A^{1/2}z(t), A^{3/2}y(t))_H \leq C \|A^{1/2}z(t)\|_H \|A^{3/2}y(t)\|_H \\ & \leq \frac{C}{\delta} \|A^{1/2}z(t)\|_H^2 + \delta \|A^{3/2}y(t)\|_H^2 \leq \frac{C}{\delta} \|A^{3/4}z(t)\|_H^{4/3} \|z(t)\|_H^{2/3} + \delta \|A^{3/2}y(t)\|_H^2 \\ & \leq \delta \|A^{3/4}z(t)\|_H^2 + C_\delta \|z(t)\|_H^2 + \delta \|A^{3/2}y(t)\|_H^2, \end{aligned}$$

$$\begin{aligned}\gamma(z(t), Ay(t))_H &\leq \gamma \|z(t)\|_H \|Ay(t)\|_H \leq \|Ay(t)\|_H^2 + \gamma^2 \|z(t)\|_H^2 \\ &\leq \delta \|A^{3/2}y(t)\|_H^2 + C_\delta \|y(t)\|_H^2 + C \|z(t)\|_H^2,\end{aligned}$$

$$\begin{aligned}\gamma(Ay(t), A^{1/2}z(t))_H &\leq \gamma \|Ay(t)\|_H \|A^{1/2}z(t)\|_H \leq \frac{C}{\delta} \|Ay(t)\|_H^2 + \delta \|A^{1/2}z(t)\|_H^2 \\ &\leq \delta \|A^{3/2}y(t)\|_H^2 + C_\delta \|y(t)\|_H^2 + C_1\delta \|A^{3/4}z(t)\|_H^2,\end{aligned}$$

$$\gamma(y(t), A^{1/2}z(t))_H \leq \delta \|A^{1/2}z(t)\|_H^2 + C_\delta \|y(t)\|_H^2 \leq C_1\delta \|A^{3/4}z(t)\|_H^2 + C_\delta \|y(t)\|_H^2,$$

$$(z(t), A^{1/2}z(t))_H \leq \delta \|A^{1/2}z(t)\|_H^2 + C_\delta \|z(t)\|_H^2 \leq C_1\delta \|A^{3/4}z(t)\|_H^2 + C_\delta \|z(t)\|_H^2,$$

$$\int_\Omega 1_\omega v(t) Ay(t) dx \leq C \|v(t)\|_H \|Ay(t)\|_H \leq C_\delta \|v(t)\|_H^2 + \delta \|A^{3/2}y(t)\|_H^2,$$

$$\int_\Omega 1_\omega u(t) A^{1/2}z(t) dx \leq C \|u(t)\|_H \|A^{1/2}z(t)\|_H \leq C_\delta \|u(t)\|_H^2 + C_1\delta \|A^{3/4}z(t)\|_H^2.$$

Plugging all these in (2.45), choosing δ small enough and recalling (2.28), we obtain

$$\begin{aligned}&\frac{1}{2} \frac{d}{dt} \left(\|A^{1/2}y(t)\|_H^2 + \|A^{1/4}z(t)\|_H^2 \right) + \frac{\nu}{2} \|A^{3/2}y(t)\|_H^2 + \frac{1}{2} \|A^{3/4}z(t)\|_H^2 \\ &\leq C(\|y(t)\|_H^2 + \|z(t)\|_H^2 + \|u(t)\|_H^2 + \|v(t)\|_H^2) \\ &\leq C \left\{ e^{-kt} (\|y^0\|_H^2 + \|z^0\|_H^2) + \|u(t)\|_H^2 + \|v(t)\|_H^2 \right\}.\end{aligned}$$

Integrating in time and using (2.29) we get

$$\begin{aligned}&\|A^{1/2}y(t)\|_H^2 + \|A^{1/4}z(t)\|_H^2 \\ &+ C_\nu \int_0^t \left(\|A^{3/2}y(s)\|_H^2 + \|A^{3/4}z(s)\|_H^2 + \|u(s)\|_H^2 + \|v(s)\|_H^2 \right) ds \\ &\leq \|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2 + \frac{C}{k} (1 - e^{-kt}) (\|y^0\|_H^2 + \|z^0\|_H^2) + C(\|y^0\|_H^2 + \|z^0\|_H^2) \\ &\leq C \left(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2 \right), \text{ for all } t > 0,\end{aligned} \tag{2.46}$$

where $C_\nu = \min\{\nu, 1\}$ and C denotes constants depending on the problem parameters ν, γ, F_l . From here we deduce that

$$\begin{aligned}&\int_0^\infty \left(\|A^{3/2}y(t)\|_H^2 + \|A^{3/4}z(t)\|_H^2 + \|u(t)\|_H^2 + \|v(t)\|_H^2 \right) dt \\ &\leq C(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2) \leq c_2(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2),\end{aligned} \tag{2.47}$$

where c_2 depends on the problem parameters. This is the right inequality in (2.42).

Now, we take in (2.40) a minimizing sequence $\{W^n\}_{n \geq 1}$, $W^n = (w_1^n, \dots, w_N^n)$ such that

$$(u_n(t), v_n(t)) = \sum_{j=1}^N w_j^n(t) (\varphi_j, \psi_j).$$

We can assume that

$$d \leq \frac{1}{2} \int_0^\infty \left(\|A^{3/2}y_n(t)\|_H^2 + \|A^{3/4}z_n(t)\|_H^2 + \|W^n(t)\|_{\mathbb{R}^N}^2 \right) dt \leq d + \frac{1}{n}, \quad (2.48)$$

where d is the positive infimum of $J(W)$ in (2.40) and (y_n, z_n) is the solution to (2.25)-(2.26) corresponding to W^n . By (2.48) we have a subsequence $\{n \rightarrow \infty\}$ such that

$$w_j^n \rightarrow w_j^* \text{ weakly in } L^2(\mathbb{R}^+), \quad j = 1, \dots, N,$$

$$y_n \rightarrow y^* \text{ weakly in } L^2(\mathbb{R}^+; D(A^{3/2})),$$

$$z_n \rightarrow z^* \text{ weakly in } L^2(\mathbb{R}^+; D(A^{3/4})).$$

Also, by (2.25) we have

$$\frac{d}{dt}(y_n, z_n) \rightarrow \frac{d}{dt}(y, z) \text{ weakly in } L^2(\mathbb{R}^+; \mathcal{V}'),$$

where \mathcal{V}' is defined in (2.16). Since $(u_n(t), v_n(t)) = \sum_{j=1}^N w_j^n(t)(\varphi_j, \psi_j)$ we have

$$(u_n, v_n) \rightarrow (u^*, v^*) = \sum_{j=1}^N w_j^*(t)(\varphi_j, \psi_j) \text{ weakly in } L^2(\mathbb{R}^+; H \times H).$$

Thus, (y^*, z^*) solves the system (2.25)-(2.26) corresponding to $W^* := (w_1^*, \dots, w_N^*)$. Moreover, passing to the limit in (2.48) we get on the basis of the weakly lower semicontinuity of J that $J(W^*) = d$.

The uniqueness follows by the fact that J is strictly convex and the state system is linear. Moreover, by (2.45) we can write

$$\begin{aligned} & \int_0^t \left(\nu \|A^{3/2}y(s)\|_H^2 + \|A^{3/4}z(s)\|_H^2 \right) ds \\ &= \frac{1}{2} \left(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2 \right) - \frac{1}{2} \left(\|A^{1/2}y(t)\|_H^2 + \|A^{1/4}z(t)\|_H^2 \right) \\ & \quad - (F_l - 2\nu) \int_0^t (Ay(s), Ay(s))_H ds + (F_l - \nu) \int_0^t (y(s), Ay(s))_H ds \\ & \quad + \gamma \int_0^t (Az(s), Ay(s))_H ds \\ & \quad - \gamma \int_0^t (z(s), Ay(s))_H ds + \gamma \int_0^t (Ay(s), A^{1/2}z(s))_H ds - \gamma \int_0^t (y(s), A^{1/2}z(s))_H ds \\ & \quad + \int_0^t (z(s), A^{1/2}z(s))_H ds + \int_0^t \int_\Omega 1_\omega^* v(s) Ay(s) dx dt + \int_0^t \int_\Omega 1_\omega^* u(s) A^{1/2} z(s) dx ds. \end{aligned}$$

We are going to derive a basic inequality by arguing as we did for all terms on the right-hand side in (2.45) in order to get (2.46), but suitably changing δ and C_δ in the use of the Young inequality. For instance, we have

$$\begin{aligned} |(F_l - 2\nu)(Ay(t), Ay(t))_H| &\leq |F_l - 2\nu| \|Ay(t)\|_H^2 \leq C \left(\|A^{3/2}y(t)\|_H^{2/3} \|A^0y(t)\|_H^{1/3} \right)^2 \\ &\leq C_\delta \|A^{3/2}y(t)\|_H^2 + \delta \|y(t)\|_H^2, \end{aligned}$$

which implies

$$-(F_l - 2\nu)(Ay(t), Ay(t))_H \geq -C_\delta \|A^{3/2}y(t)\|_H^2 - \delta \|y(t)\|_H^2.$$

By treating all the terms in the same way we arrive at

$$\begin{aligned} & \int_0^t \left(\nu \|A^{3/2}y(s)\|_H^2 + \|A^{3/4}z(s)\|_H^2 \right) ds \\ & \geq \frac{1}{2} \left(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2 \right) - \frac{1}{2} \left(\|A^{1/2}y(t)\|_H^2 + \|A^{1/4}z(t)\|_H^2 \right) \\ & \quad - C_\delta \int_0^t \left(\|A^{3/2}y(s)\|_H^2 + \|A^{3/4}z(s)\|_H^2 \right) ds - C_1 \delta \int_0^t (\|y(s)\|_H^2 + \|z(s)\|_H^2) ds \\ & \quad - \delta \int_0^t (\|u(s)\|_H^2 + \|v(s)\|_H^2) ds, \end{aligned}$$

where $C_1 > 0$ could be computed. By relying on (2.28) and (2.29) we obtain

$$\begin{aligned} & (C_\delta + \max\{\nu, 1\}) \int_0^t \left(\|A^{3/2}y(s)\|_H^2 + \|A^{3/4}z(s)\|_H^2 \right) ds \\ & \geq \frac{1}{2} \left(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2 \right) - \frac{1}{2} \left(\|A^{1/2}y(t)\|_H^2 + \|A^{1/4}z(t)\|_H^2 \right) \\ & \quad - 2C_1 \delta C_P^2 \int_0^t e^{-2ks} ds (\|y^0\|_H^2 + \|z^0\|_H^2) - 2\delta C_P^2 (\|y^0\|_H^2 + \|z^0\|_H^2). \end{aligned}$$

Computing the integral, using (1.43) and choosing δ small enough we get

$$\begin{aligned} & (C + \max\{\nu, 1\}) \int_0^t \left(\|A^{3/2}y(s)\|_H^2 + \|A^{3/4}z(s)\|_H^2 \right) ds \\ & \geq \frac{1}{4} \left(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2 \right) - \frac{1}{2} \left(\|A^{1/2}y(t)\|_H^2 + \|A^{1/4}z(t)\|_H^2 \right). \end{aligned} \tag{2.49}$$

Since the last term on the right-hand side is a continuous L^1 function, one can take a sequence $t_j \nearrow \infty$ such that

$$\|A^{1/2}y(t_j)\|_H^2 + \|A^{1/4}z(t_j)\|_H^2 \rightarrow 0.$$

Passing to the limit in (2.49) along such a sequence we obtain

$$\int_0^\infty \left(\|A^{3/2}y(s)\|_H^2 + \|A^{3/4}z(s)\|_H^2 \right) ds \geq c_1 \left(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2 \right),$$

where $c_1 > 0$ depends only on the problem parameters and $\|\varphi_\infty\|_{L^2(\Omega)}$.

This relation written for the optimal pair $(W^*, (y^*, z^*))$ implies that

$$\begin{aligned} \Phi(y^0, z^0) &= \frac{1}{2} \int_0^\infty \left(\|A^{3/2}y^*(t)\|_H^2 + \|A^{3/4}z^*(t)\|_H^2 + \|W^*(t)\|_{\mathbb{R}^N}^2 \right) dt \\ &\geq c_1 \left(\|A^{1/2}y^0\|_H^2 + \|A^{1/4}z^0\|_H^2 \right), \end{aligned}$$

that is the left inequality in (2.42).

Relation (2.47), valid also for the optimal pair, leads to the right-hand side of (2.42).

The next calculation will be done in view of proving (2.43).

We recall (2.44) and multiply (2.25) by $(A^2y, \alpha Az)$ scalarly in $H \times H$, with α a positive number that will be specified later. We obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|Ay(t)\|_H^2 + \alpha \|A^{1/2}z(t)\|_H^2 \right) + \nu \|A^2y(t)\|_H^2 + \alpha \|Az(t)\|_H^2 \\ = & -(F_l - 2\nu)(Ay(t), A^2y(t))_H + (F_l - \nu)(y(t), A^2y(t))_H + \gamma(Az(t), A^2y(t))_H \\ & - \gamma(z(t), A^2y(t))_H + \alpha(z(t), Az(t))_H + \alpha\gamma(Ay(t), Az(t))_H - \alpha\gamma(y(t), Az(t))_H \\ & + \int_{\Omega} 1_{\omega}^* v(t) A^2y(t) dx + \alpha \int_{\Omega} 1_{\omega}^* u(t) Az(t) dx. \end{aligned} \quad (2.50)$$

As previously, we have

$$\begin{aligned} -(F_l - 2\nu)(Ay(t), A^2y(t))_H & \leq \delta \|A^2y(t)\|_H^2 + C_{\delta} \|Ay(t)\|_H^2 \\ & \leq \delta \|A^2y(t)\|_H^2 + \delta \|A^{3/2}y(t)\|_H^2 + C_{\delta} \|y(t)\|_H^2, \end{aligned}$$

$$(F_l - \nu)(y(t), A^2y(t))_H \leq \delta \|A^2y(t)\|_H^2 + C_{\delta} \|y(t)\|_H^2,$$

$$\gamma(Az(t), A^2y(t))_H \leq \delta \|A^2y(t)\|_H^2 + C_{\delta} \|Az(t)\|_H^2,$$

$$-\gamma(z(t), A^2y(t))_H \leq \delta \|A^2y(t)\|_H^2 + C_{\delta} \|z(t)\|_H^2,$$

$$\alpha(z(t), Az(t))_H \leq \frac{\alpha}{8} \|Az(t)\|_H^2 + C_{\alpha} \|z(t)\|_H^2,$$

$$\begin{aligned} \alpha\gamma(Ay(t), Az(t))_H & \leq 8\gamma^2\alpha \|Ay(t)\|_H^2 + \frac{\alpha}{8} \|Az(t)\|_H^2 \\ & \leq \delta \|A^{3/2}y(t)\|_H^2 + C_{\delta}\alpha^2 \|y(t)\|_H^2 + \frac{\alpha}{8} \|Az(t)\|_H^2, \end{aligned}$$

$$-\alpha\gamma(y(t), Az(t))_H \leq \frac{\alpha}{8} \|Az(t)\|_H^2 + C_{\alpha} \|y(t)\|_H^2,$$

$$\int_{\Omega} 1_{\omega}^* v(t) A^2y(t) dx \leq \delta \|A^2y(t)\|_H^2 + C_{\delta} \|v(t)\|_H^2,$$

$$\int_{\Omega} 1_{\omega}^* u(t) Az(t) dx \leq \frac{\alpha}{8} \|Az(t)\|_H^2 + C \|u(t)\|_H^2.$$

Plugging all these relations in (2.50), using $\|A^{3/2}y(t)\|_H^2 \leq C \|A^2y(t)\|_H^2$ and choosing δ small enough we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|Ay(t)\|_H^2 + \alpha \|A^{1/2}z(t)\|_H^2 \right) + \frac{\nu}{2} \|A^2y(t)\|_H^2 + \left(\frac{\alpha}{2} - C_2 \right) \|Az(t)\|_H^2 \\ \leq & C_{\alpha} (\|y(t)\|_H^2 + \|z(t)\|_H^2 + \|v(t)\|_H^2 + \|u(t)\|_H^2), \end{aligned}$$

where C_2 can be computed and depends only the system parameters, and C_α depends on α , in addition. We choose, for instance, $\alpha = 4C_2$, integrate from 0 to t , use (2.28)-(2.29) in order to find

$$\begin{aligned} & \|Ay(t)\|_H^2 + \|A^{1/2}z(t)\|_H^2 + \int_0^t \left(\|A^2y(s)\|_H^2 + \|Az(s)\|_H^2 \right) ds \\ & \leq C \left(\|Ay^0\|_H^2 + \|A^{1/2}z^0\|_H^2 \right) + CC_p^2 (\|y^0\|_H^2 + \|z^0\|_H^2). \end{aligned}$$

Writing this relation for the optimal pair we obtain (2.43), as claimed. \square

Let us point out a first consequence of Proposition 2.3. It is not difficult to check that the functional

$$\|\cdot\|_\Phi = \sqrt{\Phi} : D(A^{1/2}) \times D(A^{1/4}) \rightarrow \mathbb{R}$$

is a norm satisfying the parallelogram law. Then, (2.42) implies that $\|\cdot\|_\Phi$ is a Hilbert norm on $D(A^{1/2}) \times D(A^{1/4})$ equivalent to the natural one. In addition, Φ is a quadratic functional. Moreover, if we denote $(\cdot, \cdot)_\Phi$ the corresponding scalar product we can introduce

$$R : \Xi := D(A^{1/2}) \times D(A^{1/4}) \rightarrow \Xi' = (D(A^{1/2}) \times D(A^{1/4}))' \quad (2.51)$$

such that

$$\langle R(y^0, z^0), (Y, Z) \rangle_{\Xi', \Xi} = \frac{1}{2} ((y^0, z^0), (Y, Z))_\Phi \text{ for all } (y^0, z^0), (Y, Z) \in \Xi.$$

In fact R coincides with $2R_\Xi$, where R_Ξ is the Riesz operator associated to $\|\cdot\|_\Phi$. In particular, we have

$$\Phi(y^0, z^0) = \frac{1}{2} (R(y^0, z^0), (y^0, z^0)) \text{ for all } (y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4}). \quad (2.52)$$

Moreover, $R(y^0, z^0)$ is the Gâteaux derivative of the function Φ at (y^0, z^0) . Indeed, for any $(Y, Z) \in D(A^{1/2}) \times D(A^{1/4}) = \Xi$ we have

$$\begin{aligned} \Phi'(y^0, z^0)(Y, Z) &= \lim_{\lambda \rightarrow 0} \frac{\Phi(y^0 + \lambda Y, z^0 + \lambda Z) - \Phi(y^0, z^0)}{\lambda} \\ &= \frac{1}{2} \lim_{\lambda \rightarrow 0} \frac{\langle R(y^0 + \lambda Y, z^0 + \lambda Z), (y^0 + \lambda Y, z^0 + \lambda Z) \rangle_{\Xi', \Xi} - \langle R(y^0, z^0), (y^0, z^0) \rangle_{\Xi', \Xi}}{\lambda} \\ &= \langle R(y^0, z^0), (Y, Z) \rangle_{\Xi', \Xi}, \end{aligned}$$

hence

$$\Phi'(y^0, z^0) = R(y^0, z^0), \text{ for all } (y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4}). \quad (2.53)$$

Since Φ is coercive by (2.42) we can define the restriction of R to $H \times H$ (denoted still by R) having the domain

$$D(R) = \{(y^0, z^0) \in \Xi; R(y^0, z^0) \in H \times H\}.$$

It also turns out that R is self-adjoint. Moreover, R can be written of the form

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}. \quad (2.54)$$

We shall give more details about this in the next Proposition which also provides a representation result for the optimal solution to (2.40).

Let us recall the operators $B : \mathbb{R}^N \rightarrow H \times H$, $B^* : H \times H \rightarrow \mathbb{R}^N$, defined in (1.17) and (1.18),

$$BW = \begin{bmatrix} \sum_{i=1}^N 1_\omega^* \varphi_i w_i \\ \sum_{i=1}^N 1_\omega^* \psi_i w_i \end{bmatrix} \quad \text{for all } W = \begin{bmatrix} w_1 \\ \dots \\ w_N \end{bmatrix} \in \mathbb{R}^N,$$

and

$$B^*q = \begin{bmatrix} \int_\Omega 1_\omega^* (\varphi_1 q_1 + \psi_1 q_2) dx \\ \dots \\ \int_\Omega 1_\omega^* (\varphi_N q_1 + \psi_N q_2) dx \end{bmatrix}, \quad \text{for all } q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in H \times H.$$

Then, (2.25)-(2.26) can be rewritten as

$$\begin{aligned} \frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) &= BW(t), \quad \text{a.e. } t > 0, \\ (y(0), z(0)) &= (y^0, z^0). \end{aligned} \quad (2.55)$$

Proposition 2.4. *Let $W^* = \{w_i^*\}_{i=1}^N$ and (y^*, z^*) be optimal for problem (2.40), corresponding to $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$. Then, W^* is expressed as*

$$W^*(t) = -B^*R(y^*(t), z^*(t)), \quad \text{for all } t > 0. \quad (2.56)$$

Moreover, R has the following properties

$$\begin{aligned} 2c_1 \|(y^0, z^0)\|_{D(A^{1/2}) \times D(A^{1/4})}^2 &\leq (R(y^0, z^0), (y^0, z^0))_{H \times H} \leq 2c_2 \|(y^0, z^0)\|_{D(A^{1/2}) \times D(A^{1/4})}^2, \\ \text{for all } (y^0, z^0) &\in D(A^{1/2}) \times D(A^{1/4}), \end{aligned} \quad (2.57)$$

$$\|R(y^0, z^0)\|_{H \times H} \leq C_R \|(y^0, z^0)\|_{D(A) \times D(A^{1/2})}, \quad \text{for all } (y^0, z^0) \in D(A) \times D(A^{1/2}), \quad (2.58)$$

and satisfies the Riccati algebraic equation (1.19), that is

$$\begin{aligned} 2(R(\bar{y}, \bar{z}), \mathcal{A}(\bar{y}, \bar{z}))_{H \times H} + \|B^*R(\bar{y}, \bar{z})\|_{\mathbb{R}^N}^2 \\ = \|A^{3/2}\bar{y}\|_H^2 + \|A^{3/4}\bar{z}\|_H^2, \quad \text{for all } (\bar{y}, \bar{z}) \in D(A^{3/2}) \times D(A^{3/4}). \end{aligned} \quad (2.59)$$

Here, c_1, c_2, C_R are constants (c_1, c_2 are the same as in (2.42) and depend on the problem parameters, Ω and $\|\varphi_\infty\|_{L^2(\Omega)}$, and C_R depends only on Ω).

Proof. We organize the proof in two steps.

Step 1. Inequalities (2.57) immediately follow from (2.52) and (2.42).

Next we prove (2.58) and (2.56).

Let T be positive and arbitrary. We recall that by the dynamic programming principle (see e.g., [2], p. 104), the minimization problem (2.40) is equivalent to the following problem

$$\text{Min}_{W \in L^2(0, T; \mathbb{R}^N)} \left\{ \frac{1}{2} \int_0^T \left(\|A^{3/2}y(t)\|_H^2 + \|A^{3/4}z(t)\|_H^2 + \|W(t)\|_{\mathbb{R}^N}^2 \right) dt + \Phi(y(T), z(T)) \right\} \quad (2.60)$$

subject to (2.25)-(2.26). Thus, a solution to (2.40) is a solution to (2.60) on $(0, T)$ and conversely.

We introduce the adjoint system

$$\begin{aligned} \frac{d}{dt}(p^T, q^T)(t) - \mathcal{A}(p^T(t), q^T(t)) &= (A^3 y^*(t), A^{3/2} z^*(t)), \text{ in } (0, T) \times \Omega, \\ (p^T(T), q^T(T)) &= -R(y^*(T), z^*(T)), \text{ in } \Omega, \end{aligned} \quad (2.61)$$

by recalling that \mathcal{A} is self-adjoint. We have used (2.53) for writing the final condition at $t = T$. For the moment we indicate this solution by (p^T, q^T) , to show its dependence on T . Later, we shall prove that actually it is independent of T . By the maximum principle in (2.60), we have that

$$W^*(t) = B^*(p^T(t), q^T(t)), \text{ a.e. } t \in (0, T) \quad (2.62)$$

(see [25], p. 114; see also [2], p. 190).

For proving (2.58), let $(y^0, z^0) \in D(A) \times D(A^{1/2})$.

Since

$$R(y^*(T), z^*(T)) \in (D(A^{1/2}))' \times (D(A^{1/4}))' \subset V' \times V'$$

and

$$(A^3 y^*, A^{3/2} z^*) \in L^2(0, T; V' \times V'),$$

it follows that (2.61) has a unique solution

$$(p^T, q^T) \in L^2(0, T; H \times H) \cap C([0, T]; V' \times V') \quad (2.63)$$

(see [1], Thm. 7.1, p. 291). We shall prove that (p^T, q^T) is in $C([0, T]; H \times H)$. For the reader's convenience we give the argument, adapting some ideas from the proof in [7]. We define

$$(\tilde{p}, \tilde{q}) = \tilde{A}(p^T, q^T) \quad (2.64)$$

where \tilde{A} is the operator

$$\tilde{A} = \begin{bmatrix} A^{-1} & 0 \\ 0 & A^{-1/2} \end{bmatrix}.$$

By recalling (2.44) we see that \mathcal{A} and \tilde{A} commute. Thus, we replace (2.64) in (2.61), obtaining the system

$$\begin{aligned} \frac{d}{dt}(\tilde{p}, \tilde{q})(t) - \mathcal{A}(\tilde{p}(t), \tilde{q}(t)) &= (A^2 y^*(t), A z^*(t)), \text{ in } (0, T) \times \Omega, \\ (\tilde{p}(T), \tilde{q}(T)) &= -\tilde{A}R(y^*(T), z^*(T)), \text{ in } \Omega. \end{aligned} \quad (2.65)$$

According to (2.43), we have $(A^2 y^*, A z^*) \in L^2(0, T; H \times H)$ and by (2.63) we obtain $\tilde{A}R(y^*(T), z^*(T)) \in V \times H$. By applying a backward version of Proposition 2.1, formula (2.18) we see that system (2.65) has a unique solution

$$(\tilde{p}, \tilde{q}) \in C([0, T]; D(A) \times D(A^{1/2}))$$

and so $(p^T, q^T) \in C([0, T]; H \times H)$. Next, we prove the relation

$$R(y^0, z^0) = -(p^T(0), q^T(0)). \quad (2.66)$$

To this end, let us consider two solutions to (2.60), (W^*, y^*, z^*) and (W_1^*, y_1^*, z_1^*) , corresponding to (y^0, z^0) and (y^1, z^1) , respectively, both in $D(A) \times D(A^{1/2})$. Using the subdifferential inequality $\|v\|_X - \|v_1\|_X \leq 2(v, v - v_1)_X$, which holds in any Hilbert space X , and the relation $\Phi' = R$ we compute

$$\begin{aligned} & \Phi(y^0, z^0) - \Phi(y_1, z_1) \\ & \leq \int_0^T \{ (A^{3/2}y^*(t), A^{3/2}(y^*(t) - y_1^*(t)))_H + (A^{3/4}z^*(t), A^{3/4}(z^*(t) - z_1^*(t)))_H \} dt \\ & \quad + \int_0^T (W^*(t), W^*(t) - W_1^*(t))_{\mathbb{R}^N} dt \\ & \quad + (R(y^*(T), z^*(T)), (y^*(T) - y_1^*(T), z^*(T) - z_1^*(T)))_{H \times H}. \end{aligned} \tag{2.67}$$

It is clear that $W^* = (w_1^*, \dots, w_N^*)$ and $W_1^* = (w_{11}^*, \dots, w_{1N}^*)$, respectively. By multiplying (2.61) by $(y^*(t) - y_1^*(t), z^*(t) - z_1^*(t))$, integrating by parts and using the difference of the state equations (2.55), written for both solutions, we obtain that

$$\begin{aligned} & \frac{d}{dt} ((p^T(t), q^T(t)), (y^*(t) - y_1^*(t), z^*(t) - z_1^*(t)))_{H \times H} \\ & = (A^{3/2}y^*(t), A^{3/2}(y^*(t) - y_1^*(t)))_H + (A^{3/4}z^*(t), A^{3/4}(z^*(t) - z_1^*(t)))_H \\ & \quad + ((p^T(t), q^T(t)), BW^*(t) - BW_1^*(t))_{H \times H}, \text{ a.e. } t > 0. \end{aligned} \tag{2.68}$$

Now, we integrate (2.68) over $(0, T)$ and use the final condition in (2.61) and (2.62), to get

$$\begin{aligned} & -(R(y^*(T), z^*(T)), (y^*(T) - y_1^*(T), z^*(T) - z_1^*(T)))_{H \times H} \\ & - ((p^T(0), q^T(0)), (y^*(0) - y_1^*(0), z^*(0) - z_1^*(0)))_{H \times H} \\ & = \int_0^T \{ (A^{3/2}y^*(t), A^{3/2}(y^*(t) - y_1^*(t)))_H + (A^{3/4}z^*(t), A^{3/4}(z^*(t) - z_1^*(t)))_H \} dt \\ & \quad + \int_0^T (W^*(t), W^*(t) - W_1^*(t))_{\mathbb{R}^N} dt, \end{aligned}$$

whence by (2.67) we finally obtain

$$\Phi(y^0, z^0) - \Phi(y^1, z^1) \leq -((p^T(0), q^T(0)), (y^0 - y^1, z^0 - z^1))_{H \times H}. \tag{2.69}$$

This implies that

$$-(p^T(0), q^T(0)) \in \partial\Phi(y^0, z^0).$$

Since, as seen earlier, Φ is differentiable on $D(A^{1/2}) \times D(A^{1/4})$ it follows that

$$-(p^T(0), q^T(0)) = \Phi'(y^0, z^0) = R(y^0, z^0),$$

as claimed in (2.66). This implies, since we have proved that $(p^T, q^T) \in C([0, \infty); H \times H)$, that $(p^T(0), q^T(0)) \in H \times H$ and so

$$R(y^0, z^0) \in H \times H \text{ for all } (y^0, z^0) \in D(A) \times D(A^{1/2}). \tag{2.70}$$

On the other hand, one can easily see that R is a linear closed operator from $D(A) \times D(A^{1/2})$ to $H \times H$, and so by the closed graph theorem we conclude that it is continuous (see e.g. [9], Thm. 2.9, p. 37), that is $R \in \mathcal{L}(D(A) \times D(A^{1/2}); H \times H)$, as claimed by (2.58).

We define the restriction of R to $H \times H$, still denoted by R . Thus, its domain contains $D(A) \times D(A^{1/2})$.

Now, we resume (2.62) which extends by the continuity (2.63) at $t = T$, in V' .

$$W^*(T) = B^*(p^T(T), q^T(T)). \quad (2.71)$$

Moreover, since $(y^*(t), z^*(t)) \in D(A) \times D(A^{1/2})$ for all $t \geq 0$, by (2.43), we have by (2.70) that $R(y^*(t), z^*(t)) \in H \times H$ for all $t \geq 0$. In particular, this is true for $t = T$ and so using the final condition in (2.61) we get

$$(p^T(T), q^T(T)) = -R(y^*(T), z^*(T)) \in H \times H. \quad (2.72)$$

This relation combined with (2.71) implies

$$W^*(T) = -B^*R(y^*(T), z^*(T))$$

where T is arbitrary. Therefore, it can be written for any t , as in (2.56), as claimed.

By (2.56) and by the definition (1.18) and (2.27) we can write

$$w_j = -(B^*R(y^*(t), z^*(t)))_j = - \int_{\Omega} 1_{\omega}^* (\varphi_j R_1(y^*(t), z^*(t)) + \psi_j R_2(y^*(t), z^*(t))) dx \quad (2.73)$$

and by (2.54) we get

$$R_1(y^*(t), z^*(t)) = R_{11}y^*(t) + R_{12}z^*(t), \quad R_2(y^*(t), z^*(t)) = R_{21}y^*(t) + R_{22}z^*(t). \quad (2.74)$$

In particular,

$$\begin{aligned} (v^*, u^*) &= \left(\sum_{j=1}^N \varphi_j w_j, \sum_{j=1}^N \psi_j w_j \right) \\ &= \left(- \sum_{j=1}^N \varphi_j (B^*R(y^*(t), z^*(t)))_j, - \sum_{j=1}^N \psi_j (B^*R(y^*(t), z^*(t)))_j \right) \end{aligned}$$

which implies by (1.18) the representation

$$\begin{aligned} v^*(t, x) &= - \sum_{j=1}^N \varphi_j(x) (B^*R(y^*(t), z^*(t)))_j \\ &= - \sum_{j=1}^N \varphi_j(x) \int_{\Omega} 1_{\omega}^* (\varphi_j R_1(y^*(t), z^*(t)) + \psi_j R_2(y^*(t), z^*(t))) (\xi) d\xi, \end{aligned} \quad (2.75)$$

$$\begin{aligned} u^*(t, x) &= - \sum_{j=1}^N \psi_j(x) (B^*R(y^*(t), z^*(t)))_j \\ &= - \sum_{j=1}^N \psi_j(x) \int_{\Omega} 1_{\omega}^* (\varphi_j R_1(y^*(t), z^*(t)) + \psi_j R_2(y^*(t), z^*(t))) (\xi) d\xi. \end{aligned} \quad (2.76)$$

Finally, it follows by (2.56) that

$$1_\omega^* U(t) = 1_\omega^*(v^*(t), u^*(t)) = -BB^*R(y^*(t), z^*(t)). \quad (2.77)$$

Step 2. We pass now to the proof of (2.59). It is enough to consider $(y^0, z^0) \in D(A) \times D(A^{1/2})$. Since (W^*, y^*, z^*) is the solution to both (2.40) and (2.60) written with $T = t$ where $t \geq 0$ is arbitrary and the minimum is $\Phi(y^0, z^0)$ we can write

$$\begin{aligned} \Phi(y^0, z^0) &= \frac{1}{2} \int_0^t \left(\|A^{3/2}y^*(s)\|_H^2 + \|A^{3/4}z^*(s)\|_H^2 + \|W^*(s)\|_{\mathbb{R}^N}^2 \right) ds \\ &+ \frac{1}{2} \int_t^\infty \left(\|A^{3/2}y^*(s)\|_H^2 + \|A^{3/4}z^*(s)\|_H^2 + \|W^*(s)\|_{\mathbb{R}^N}^2 \right) ds \\ &= \Phi(y^0, z^0) - \Phi(y^*(t), z^*(t)) + \frac{1}{2} \int_t^\infty \left(\|A^{3/2}y^*(s)\|_H^2 + \|Az^*(s)\|_H^2 + \|W^*(s)\|_{\mathbb{R}^N}^2 \right) ds \end{aligned}$$

and so

$$\Phi(y^*(t), z^*(t)) = \frac{1}{2} \int_t^\infty \left(\|A^{3/2}y^*(s)\|_H^2 + \|A^{3/4}z^*(s)\|_H^2 + \|W^*(s)\|_{\mathbb{R}^N}^2 \right) ds, \quad (2.78)$$

for any $t \geq 0$. Now, we want to differentiate (2.78) with respect to t . To this aim, we recall (2.52) and that R is symmetric. Thus,

$$\begin{aligned} \frac{d}{dt} \Phi(y^*(t), z^*(t)) &= \frac{1}{2} \frac{d}{dt} (R(y^*(t), z^*(t)), (y^*(t), z^*(t)))_{H \times H} \\ &= \left(R(y^*(t), z^*(t)), \frac{d}{dt} (y^*(t), z^*(t)) \right)_{H \times H}. \end{aligned}$$

Hence, taking into account (2.56) we obtain for a.e. $t > 0$ (since $A^{3/2}y^*(t)$ is defined only for a.e. t) that

$$\begin{aligned} &\left(R(y^*(t), z^*(t)), \frac{d}{dt} (y^*(t), z^*(t)) \right)_{H \times H} + \frac{1}{2} \left(\|A^{3/2}y^*(t)\|_H^2 + \|A^{3/4}z^*(t)\|_H^2 \right) \\ &+ \frac{1}{2} \|B^*R(y^*(t), z^*(t))\|_{\mathbb{R}^N}^2 = 0. \end{aligned} \quad (2.79)$$

Now, we come back to the system (2.55) in which the right-hand side is replaced by (2.77). This becomes a *closed loop* system with the right-hand side $-BB^*R(y^*(t), z^*(t))$.

We take into account that by (2.58)

$$\|BB^*R(y^*(t), z^*(t))\|_{H \times H} \leq C_1 \|R(y^*(t), z^*(t))\|_{H \times H} \leq C_2 \|(y^*(t), z^*(t))\|_{D(A) \times D(A^{1/2})}$$

a.e. $t > 0$, for all $(y^*(t), z^*(t)) \in D(A) \times D(A^{1/2})$.

We show that $-(\mathcal{A} + BB^*R)$ generates a C_0 -semigroup in $H \times H$, using Lemma A3 in Appendix.

In our case, we particularize $E = D(A^2) \times D(A)$, $F = H \times H$, $L = \mathcal{A}$ and $M = BB^*R$. Then, the operator $\mathcal{A} + BB^*R$ is quasi m -accretive and we have the result. Thus, for $(y^0, z^0) \in H \times H$ we have

$$\mathcal{A}(y^*(t), z^*(t)), BB^*R(y^*(t), z^*(t)), \frac{d}{dt} (y^*(t), z^*(t)) \in C((0, \infty); H \times H)$$

(see [8], p. 72, for a basic result). Then, we can replace $\frac{d}{dt}(y^*(t), z^*(t))$ from (2.55) and plug it in (2.79). On account of (2.56) we have

$$\begin{aligned} & (R(y^*(t), z^*(t)), -\mathcal{A}(y^*(t), z^*(t)))_{H \times H} + \frac{1}{2} \left(\|A^{3/2}y^*(t)\|_H^2 + \|A^{3/4}z^*(t)\|_H^2 \right) \\ & + \frac{1}{2} \|B^*R(y^*(t), z^*(t))\|_{\mathbb{R}^N}^2 = (R(y^*(t), z^*(t)), BB^*R(y^*(t), z^*(t)))_{H \times H}, \quad t \geq 0 \end{aligned}$$

which implies (2.59) (written with a generic notation $(\bar{y}, \bar{z}) \in D(A^{3/2}) \times D(A^{3/4})$), as claimed. \square

Remark 2.5. We note that the previous equation can be still written

$$\begin{aligned} & 2(R(y^*(t), z^*(t)), \mathcal{A}(y^*(t), z^*(t)))_{H \times H} + (B^*R(y^*(t), z^*(t)), B^*R(y^*(t), z^*(t)))_{\mathbb{R}^N \times \mathbb{R}^N} \\ & = (\hat{\mathcal{A}}(y^*(t), z^*(t)), (y^*(t), z^*(t)))_{H \times H}, \end{aligned}$$

where $\hat{\mathcal{A}}: D(A^{3/2}) \times D(A^{3/4}) \rightarrow (D(A^{3/2}) \times D(A^{3/4}))'$ is defined by

$$\begin{aligned} & \left\langle \hat{\mathcal{A}}(y, z), (\psi_1, \psi_2) \right\rangle_{(D(A^{3/2}) \times D(A^{3/4}))', D(A^{3/2}) \times D(A^{3/4})} \\ & = (A^{3/2}y, A^{3/2}\psi_1)_{H \times H} + (A^{3/4}z, A^{3/4}\psi_2)_{H \times H} \end{aligned}$$

for all $(\psi_1, \psi_2) \in D(A^{3/2}) \times D(A^{3/4})$. By $\hat{\mathcal{A}}$ we still denote its restriction to $H \times H$,

$$\hat{\mathcal{A}} = \begin{bmatrix} A^3 & 0 \\ 0 & A^{3/2} \end{bmatrix}$$

with the domain $D(\hat{\mathcal{A}}) = \{(y, z) \in D(A^{3/2}) \times D(A^{3/4}); \hat{\mathcal{A}}(y, z) \in H \times H\}$.

Since

$$\begin{aligned} ((R(y, z), \mathcal{A}(y, z)))_{H \times H} &= (\mathcal{A}R(y, z), (y, z))_{H \times H}, \\ ((\mathcal{A}(y, z), R(y, z)))_{H \times H} &= ((y, z), \mathcal{A}R(y, z))_{H \times H} \end{aligned}$$

we get $\mathcal{A}R = R\mathcal{A}$, thus

$$2R\mathcal{A}(y^*(t), z^*(t)) + RBB^*R(y^*(t), z^*(t)) = \hat{\mathcal{A}}(y^*(t), z^*(t)),$$

for all $t > 0$. Letting $t \rightarrow 0$ we obtain

$$2R\mathcal{A}(y^0, z^0) + RBB^*R(y^0, z^0) = \hat{\mathcal{A}}(y^0, z^0),$$

for all $(y^0, z^0) \in D(A) \times D(A^{1/2})$ and so the Riccati equation takes the form

$$2R\mathcal{A} + RBB^*R = \hat{\mathcal{A}}. \quad (2.80)$$

Remark 2.6. Just as a remark, we observe that the linear system is exponentially stabilized to $(0, 0)$ by the feedback controller just constructed. To sustain this assertion we recall a generalization of Datko's result (see Lemma A4 in Appendix, see also [28], p.116). In our

case, the operator $\mathcal{D} = -(\mathcal{A} + BB^*R)$ generates a C_0 -semigroup in $H \times H$ and, as seen earlier, equation (2.25)

$$\frac{d}{dt}(y(t), z(t)) + (\mathcal{A} + BB^*R)(y(t), z(t)) = 0, \quad t \geq 0$$

has the solution $(y(t), z(t))$ with the property

$$\begin{aligned} \int_0^\infty (\|y(t)\|_H^2 + \|z(t)\|_H^2) dt &\leq C \int_0^\infty (\|A^{3/2}y(t)\|_H^2 + \|A^{3/4}z(t)\|_H^2) dt \\ &\leq C (\|y^0\|_{D(A^{1/2})}^2 + \|z^0\|_{D(A^{1/4})}^2) < \infty. \end{aligned}$$

Hence

$$\|y(t)\|_H^2 + \|z(t)\|_H^2 \leq Ce^{-\kappa t} (\|y^0\|_{D(A^{1/2})}^2 + \|z^0\|_{D(A^{1/4})}^2), \quad \text{for all } t \geq 0,$$

as claimed.

3 Feedback stabilization of the nonlinear system

We recall that B and B^* are defined by (1.17) and (1.18), and that $R_1 = (R_{11}, R_{12})$, $R_2 = (R_{21}, R_{22})$ are given by (2.74).

In this section we shall deal with the nonlinear system (2.5)-(2.8) in which the right-hand side $(1_\omega^*v, 1_\omega^*u)$ is replaced by the feedback controller determined in the previous section, that is, we replace $1_\omega^*U(t) = (1_\omega^*v(t), 1_\omega^*u(t))$ by $-BB^*R(y(t), z(t))$. As (2.14)-(2.15) is the abstract form of (2.10)-(2.13), the abstract form of the nonlinear system (2.5)-(2.8) with replaced right-hand side reads

$$\begin{aligned} \frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) &= \mathcal{G}(y(t)) - BB^*R(y(t), z(t)), \quad \text{a.e. } t > 0, \\ (y(0), z(0)) &= (y_0, z_0), \end{aligned} \quad (3.1)$$

where (y_0, z_0) is fixed now by (1.34), $\mathcal{G}(y(t)) = (G(y(t)), 0)$ and

$$G(y) = \Delta F_r(y) + \Delta(g(x)y). \quad (3.2)$$

We recall that F_r is the rest of second order of the Taylor expansion of $F'(y + \varphi_\infty)$ and g is defined by (2.4). Using the rest in integral form we have

$$F_r(y) = y^2 \int_0^1 (1-s) F'''(\varphi_\infty + sy) ds = y^3 + 3\varphi_\infty y^2, \quad (3.3)$$

and assuming that all operations make sense (this will be checked later) we get

$$G(y) = \sum_{j=1}^7 I_j, \quad (3.4)$$

$$\begin{aligned} I_1(y) &= 3y^2 \Delta y, \quad I_2(y) = 6y |\nabla y|^2, \quad I_3(y) = 12y \nabla y \cdot \nabla \varphi_\infty, \quad I_4(y) = 3y^2 \Delta \varphi_\infty, \\ I_5(y) &= 6\varphi_\infty y \Delta y, \quad I_6(y) = 6\varphi_\infty |\nabla y|^2, \\ I_7(y) &= \Delta(gy) = g \Delta y + y \Delta g + 2 \nabla y \cdot \nabla g. \end{aligned}$$

As usually, in the sequel, φ_∞ is the first component of a stationary solution of the uncontrolled system (1.28)-(1.30). We set

$$\chi_\infty := \|\nabla \varphi_\infty\|_\infty + \|\Delta \varphi_\infty\|_\infty. \quad (3.5)$$

Theorem 3.1. *There exists $\chi_0 > 0$ (depending on the problem parameters, the domain and $\|\varphi_\infty\|_\infty$) such that the following holds true. If $\chi_\infty \leq \chi_0$, there exists ρ such that for all pairs $(y_0, z_0) \in D(A^{1/2}) \times D(A^{1/4})$ with*

$$\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})} \leq \rho, \quad (3.6)$$

the closed loop system (3.1) has a unique solution

$$(y, z) \in C([0, \infty); H \times H) \cap L^2(0, \infty; D(A^{3/2}) \times D(A^{3/4})) \cap W^{1,2}(0, \infty; (D(A^{1/2}) \times D(A^{1/4}))'), \quad (3.7)$$

which is exponentially stable, that is

$$\|y(t)\|_{D(A^{1/2})} + \|z(t)\|_{D(A^{1/4})} \leq C_P e^{-kt} (\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})}), \quad (3.8)$$

for some positive constants k and C_P .

In the previous relations the positive constants k and C_P depend on Ω , the problem parameters and $\|\varphi_\infty\|_\infty$. In addition, C_P depends on the full norm $\|\varphi_\infty\|_{2,\infty}$.

Proof. The proof of this theorem will be done in three steps regarding the existence, uniqueness and stabilization. First, existence and uniqueness are proved on every interval $[0, T]$ and then they will be extended to the whole $[0, \infty)$.

Step 1. Existence of the solution is proved on every interval $[0, T]$ by the Schauder fixed point theorem. Let $(y_0, z_0) \in D(A^{1/2}) \times D(A^{1/4})$.

Let r be positive and bounded by a constant which will be specified later. For $T > 0$ arbitrary, but fixed, we introduce the set

$$S_T = \left\{ (y, z) \in L^2(0, T; H \times H); \sup_{t \in (0, T)} \left(\|y(t)\|_{D(A^{1/2})}^2 + \|z(t)\|_{D(A^{1/4})}^2 \right) + \int_0^T \left(\|A^{3/2} y(t)\|_H^2 + \|A^{3/4} z(t)\|_H^2 \right) dt \leq r^2 \right\}. \quad (3.9)$$

Let $0 < \varepsilon < 1/4$. Clearly, S_T is a convex closed subset of $L^2(0, T; D(A^{3/2-\varepsilon}) \times H)$.

We fix $(\bar{y}, \bar{z}) \in S_T$ and consider the Cauchy problem

$$\begin{aligned} \frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) + BB^* R(y(t), z(t)) &= \mathcal{G}(\bar{y}(t)), \text{ a.e. } t \in (0, T), \\ (y(0), z(0)) &= (y_0, z_0). \end{aligned} \quad (3.10)$$

We prove that such a problem is well-posed and define $\Psi_T : S_T \rightarrow L^2(0, T; D(A^{3/2-\varepsilon}) \times H)$ by $\Psi_T(\bar{y}, \bar{z}) = (y, z)$ the solution to (3.10). We shall prove that:

- i) $\Psi_T(S_T) \subset S_T$ provided that r is well chosen;
- ii) $\Psi_T(S_T)$ is relatively compact in $L^2(0, T; D(A^{3/2-\varepsilon}) \times H)$;
- iii) Ψ_T is continuous in the $L^2(0, T; D(A^{3/2-\varepsilon}) \times H)$ norm.

i) We assert that $G(\bar{y}) \in L^2(0, T; H \times H)$, relying on the calculation which shall be made a little later, concluded by (3.28). Then, we recall that $\mathcal{A} + BB^*R$ is m -accretive in $H \times H$, as proved in Proposition 2.4, second part, and so, for $(y_0, z_0) \in D(A^{1/2}) \times D(A^{1/4}) \subset H \times H$ it follows that the Cauchy problem (3.10) has a unique solution

$$(y, z) \in L^2(\delta, T; D(\mathcal{A})) \quad (3.11)$$

with $\delta > 0$ arbitrary (see the last part of Proposition 2.4), which implies that $\mathcal{A}(y(t), z(t)) \in H \times H$ a.e. $t > 0$ (see [8], p. 72). Therefore, also (recall (2.44)) $y(t) \in D(A^2)$ and $z(t) \in D(A)$ a.e. Moreover, by Proposition 2.1 we have

$$(y, z) \in C([0, T]; H \times H) \cap L^2(0, T; D(A) \times D(A^{1/2})) \cap W^{1,2}([0, T]; (D(A) \times D(A^{1/2}))'). \quad (3.12)$$

In particular, $(y(t), z(t)) \in D(A) \times D(A^{1/2})$ a.e. $t \in (0, T)$ and so $R(y(t), z(t)) \in H \times H$ a.e. $t \in (0, T)$.

Next, we have to prove that $(y, z) \in S_T$ provided that r is well chosen. To this end we multiply (3.10) by $R(y(t), z(t)) \in H \times H$ scalarly in $H \times H$ and get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (R(y(t), z(t)), (y(t), z(t)))_{H \times H} + (\mathcal{A}(y(t), z(t)), R(y(t), z(t)))_{H \times H} \\ &= - \|B^*R(y(t), z(t))\|_{\mathbb{R}^N}^2 + (\mathcal{G}(\bar{y}(t)), R(y(t), z(t)))_{H \times H}, \text{ a.e. } t > 0. \end{aligned}$$

Therefore, using the Riccati equation (2.59), in the form

$$2(\mathcal{A}(y(t), z(t)), R(y(t), z(t)))_{H \times H} + \|B^*R(y(t), z(t))\|_{\mathbb{R}^N}^2 = \|A^{3/2}y(t)\|_H^2 + \|A^{3/4}z(t)\|_H^2$$

and recalling (2.58) and (1.43) we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (R(y(t), z(t)), (y(t), z(t)))_{H \times H} \\ &+ \frac{1}{2} \left(\|A^{3/2}y(t)\|_H^2 + \|A^{3/4}z(t)\|_H^2 + \|B^*R(y(t), z(t))\|_{\mathbb{R}^N}^2 \right) \\ &\leq \|G(\bar{y}(t))\|_{H \times H} \|R(y(t), z(t))\|_{H \times H} \leq C_R \|G(\bar{y}(t))\|_H \|(y(t), z(t))\|_{D(A) \times D(A^{1/2})} \\ &\leq CC_R \|G(\bar{y}(t))\|_H \left(\|A^{3/2}y(t)\|_H^2 + \|A^{3/4}z(t)\|_H^2 \right)^{1/2}, \text{ a.e. } t \in (0, T), \end{aligned}$$

with C_R from (2.58) and C depending on Ω . Integrating over $(0, t)$ and using then Young's inequality and (2.57) we successively get

$$\begin{aligned} & c'_1 (\|y(t)\|_{D(A^{1/2})}^2 + \|z(t)\|_{D(A^{1/4})}^2) + \int_0^t \left(\|A^{3/2}y(s)\|_H^2 + \|A^{3/4}z(s)\|_H^2 \right) ds \\ &\leq c'_2 (\|y_0\|_{D(A^{1/2})}^2 + \|z_0\|_{D(A^{1/4})}^2) + C'_R \int_0^t \|G(\bar{y}(s))\|_H^2 ds, \quad t \in (0, T), \end{aligned} \quad (3.13)$$

where c'_1, c'_2, C'_R are proportional to c_1, c_2, C_R^2 . Now let us prove that $G(\bar{y}) \in L^2(0, T; H)$ and for that we shall estimate each term I_j in (3.4) and show that $I_j(\bar{y}) \in L^2(0, T; H)$, for

$j = 1, \dots, 7$. In the computations below we shall use the interpolation inequalities (1.42)-(1.44), and (1.39). The constants we shall introduce do not depend on φ_∞ . By (3.4) we compute

$$\begin{aligned} \|I_1(\bar{y})\|_H^2 &= C \|\bar{y}^2 \Delta \bar{y}\|_H^2 = C \int_\Omega \bar{y}^4 (\Delta \bar{y})^2 dx \leq C \left(\int_\Omega \bar{y}^8 dx \right)^{1/2} \left(\int_\Omega (\Delta \bar{y})^4 dx \right)^{1/2} \\ &= C \|\bar{y}\|_{L^8(\Omega)}^4 \|\Delta \bar{y}\|_{L^4(\Omega)}^2 \leq C \|\bar{y}\|_{H^{\alpha_1}(\Omega)}^4 \|\Delta \bar{y}\|_{H^{\alpha_2}(\Omega)}^2 \\ &\leq C \|\bar{y}\|_{H^{\alpha_1}(\Omega)}^4 \|\bar{y} - A\bar{y}\|_{H^{\alpha_2}(\Omega)}^2 \leq C \|A^0 \bar{y}\|_{H^{\alpha_1}(\Omega)}^4 \|A\bar{y}\|_{H^{\alpha_2}(\Omega)}^2 \\ &\leq C \|A^{\alpha_1/2} \bar{y}\|_H^4 \|A^{1+\alpha_2/2} \bar{y}\|_H^2, \end{aligned}$$

where $\alpha_1 \geq \frac{9}{8}$ and $\alpha_2 \geq \frac{3}{4}$. Furthermore, we have

$$\begin{aligned} \|I_1(\bar{y})\|_H^2 &\leq C \left(\|A^{3/2} \bar{y}\|_H^{(\alpha_1-1)/2} \|A^{1/2} \bar{y}\|_H^{(3-\alpha_1)/2} \right)^4 \left(\|A^{3/2} \bar{y}\|_H^{(\alpha_2+1)/2} \|A^{1/2} \bar{y}\|_H^{(1-\alpha_2)/2} \right)^2 \\ &= C \|A^{3/2} \bar{y}\|_H^{2\alpha_1+\alpha_2-1} \|A^{1/2} \bar{y}\|_H^{7-(2\alpha_1+\alpha_2)}, \quad \alpha_1 \geq \frac{9}{8}, \quad \alpha_2 \geq \frac{3}{4}. \end{aligned} \quad (3.14)$$

Since $\|A^{1/2} \bar{y}\|_H^{7-(2\alpha_1+\alpha_2)} = \|A^{1/2} \bar{y}\|_H^{3-(2\alpha_1+\alpha_2)} \|A^{1/2} \bar{y}\|_H^4 \leq C \|A^{3/2} \bar{y}\|_H^{3-(2\alpha_1+\alpha_2)} \|A^{1/2} \bar{y}\|_H^4$, we get

$$\|I_1(\bar{y})\|_H^2 \leq C \|A^{3/2} \bar{y}\|_H^2 \|A^{1/2} \bar{y}\|_H^4.$$

But $(\bar{y}, \bar{z}) \in S_T$ and therefore

$$\int_0^T \|I_1(\bar{y})\|_H^2 dt \leq Cr^4 \int_0^T \|A^{3/2} \bar{y}\|_H^2 dt \leq Cr^6. \quad (3.15)$$

For the second term we infer that

$$\begin{aligned} \|I_2(\bar{y})\|_H^2 &= C \|\bar{y} |\nabla \bar{y}|^2\|_H^2 = C \int_\Omega \bar{y}^2 |\nabla \bar{y}|^4 dx \leq C \left(\int_\Omega \bar{y}^4 dx \right)^{1/2} \left(\int_\Omega |\nabla \bar{y}|^8 dx \right)^{1/2} \\ &= C \|\bar{y}\|_{L^4(\Omega)}^2 \|\nabla \bar{y}\|_{L^8(\Omega)}^4 \leq C \|\bar{y}\|_{H^{\alpha_1}(\Omega)}^2 \|\nabla \bar{y}\|_{H^{\alpha_2}(\Omega)}^4 \\ &\leq C \|A^{\alpha_1/2} \bar{y}\|_H^2 \|A^{1/2+\alpha_2/2} \bar{y}\|_H^4 \\ &\leq C \left(\|A^{3/2} \bar{y}\|_H^{(\alpha_1-1)/2} \|A^{1/2} \bar{y}\|_H^{(3-\alpha_1)/2} \right)^2 \left(\|A^{3/2} \bar{y}\|_H^{\alpha_2/2} \|A^{1/2} \bar{y}\|_H^{(2-\alpha_2)/2} \right)^4 \\ &\leq C \|A^{3/2} \bar{y}\|_H^{\alpha_1+2\alpha_2-1} \|A^{1/2} \bar{y}\|_H^{7-(\alpha_1+2\alpha_2)}, \quad \alpha_1 \geq \frac{3}{4}, \quad \alpha_2 \geq \frac{9}{8}. \end{aligned} \quad (3.16)$$

Thus, by choosing $\alpha_1 = 3/4$ and $\alpha_2 = 9/8$, we have

$$\|I_2(\bar{y})\|_H^2 \leq C \|A^{3/2} \bar{y}\|_H^2 \|A^{1/2} \bar{y}\|_H^4. \quad (3.17)$$

For I_3 to I_6 we have the following estimates:

$$\begin{aligned} \|I_3(\bar{y})\|_H^2 &= C \|\bar{y} \nabla \bar{y} \cdot \nabla \varphi_\infty\|_H^2 \leq C \|\nabla \varphi_\infty\|_\infty^2 \|\bar{y}\|_{L^4(\Omega)}^2 \|\nabla \bar{y}\|_{L^4(\Omega)}^2 \\ &\leq C \|\nabla \varphi_\infty\|_\infty^2 \|\bar{y}\|_{H^1(\Omega)}^2 \|\bar{y}\|_{H^2(\Omega)}^2 \leq C \|\nabla \varphi_\infty\|_\infty^2 \|A^{1/2} \bar{y}\|_H^2 \|A\bar{y}\|_H^2 \\ &\leq C \|\nabla \varphi_\infty\|_\infty^2 \|A^{3/2} \bar{y}\|_H^2 \|A^{1/2} \bar{y}\|_H^2, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \|I_4(\bar{y})\|_H^2 &= C \|\bar{y}^2 \Delta \varphi_\infty\|_H^2 \leq C \|\Delta \varphi_\infty\|_\infty^2 \|\bar{y}\|_{L^4(\Omega)}^4 \leq C \|\Delta \varphi_\infty\|_\infty^2 \|\bar{y}\|_{H^1(\Omega)}^4 \quad (3.19) \\ &\leq C \|\Delta \varphi_\infty\|_\infty^2 \|A^{1/2} \bar{y}\|_H^4 \leq C \|\Delta \varphi_\infty\|_\infty^2 \|A^{3/2} \bar{y}\|_H^2 \|A^{1/2} \bar{y}\|_H^2, \end{aligned}$$

$$\begin{aligned} \|I_5(\bar{y})\|_H^2 &= C \|\varphi_\infty \bar{y} \Delta \bar{y}\|_H^2 \leq C \|\varphi_\infty\|_\infty^2 \|\bar{y}\|_{L^4(\Omega)}^2 \|\Delta \bar{y}\|_{L^4(\Omega)}^2 \quad (3.20) \\ &\leq C \|\varphi_\infty\|_\infty^2 \|\bar{y}\|_{H^1(\Omega)}^2 \|\Delta \bar{y}\|_{H^1(\Omega)}^2 \leq C \|\varphi_\infty\|_\infty^2 \|A^{1/2} \bar{y}\|_H^2 \|A^{3/2} \bar{y}\|_H^2, \end{aligned}$$

$$\begin{aligned} \|I_6(\bar{y})\|_H^2 &= C \|\varphi_\infty |\nabla \bar{y}|^2\|_H^2 \leq C \|\varphi_\infty\|_\infty^2 \|\nabla \bar{y}\|_{L^4(\Omega)}^4 \leq C \|\varphi_\infty\|_\infty^2 \|\bar{y}\|_{H^2(\Omega)}^4 \quad (3.21) \\ &\leq C \|\varphi_\infty\|_\infty^2 \|A \bar{y}\|_H^4 \leq C \|\varphi_\infty\|_\infty^2 \|A^{3/2} \bar{y}\|_H^2 \|A^{1/2} \bar{y}\|_H^2, \end{aligned}$$

and finally

$$\begin{aligned} \|I_7(\bar{y})\|_H^2 &= \|g \Delta \bar{y} + \bar{y} \Delta g + 2 \nabla \bar{y} \cdot \nabla g\|_H^2 \quad (3.22) \\ &\leq C \|g\|_{2,\infty}^2 \left(\|A \bar{y}\|_H^2 + \|\bar{y}\|_H^2 + \|A^{1/2} \bar{y}\|_H^2 \right) \leq C \|g\|_{2,\infty}^2 \|A^{3/2} \bar{y}\|_H^2. \end{aligned}$$

Moreover, by (2.4) we get

$$\begin{aligned} |g(x)| &\leq \frac{6 \|\varphi_\infty\|_\infty}{m_\Omega} \int_\Omega |\varphi_\infty(x) - \varphi_\infty(\xi)| d\xi \leq 6 \|\varphi_\infty\|_\infty \|\nabla \varphi_\infty\|_\infty d_\Omega, \\ \|g\|_\infty &\leq C \|\varphi_\infty\|_\infty \|\nabla \varphi_\infty\|_\infty, \end{aligned} \quad (3.23)$$

with d_Ω the supremum of the geodesic distance of Ω . Next,

$$\nabla g = 6 \varphi_\infty \nabla \varphi_\infty, \quad \|\nabla g\|_\infty \leq C \|\varphi_\infty\|_\infty \|\nabla \varphi_\infty\|_\infty \quad (3.24)$$

$$\Delta g = 6 \varphi_\infty \Delta \varphi_\infty + 6 |\nabla \varphi_\infty|^2, \quad \|\Delta g\|_\infty \leq C (\|\varphi_\infty\|_\infty \|\Delta \varphi_\infty\|_\infty + \|\nabla \varphi_\infty\|_\infty^2), \quad (3.25)$$

whence

$$\|g\|_{2,\infty} \leq C_\Omega \bar{g}_\infty \quad (3.26)$$

where

$$\bar{g}_\infty = \|\varphi_\infty\|_\infty \|\nabla \varphi_\infty\|_\infty + \|\varphi_\infty\|_\infty \|\Delta \varphi_\infty\|_\infty + \|\nabla \varphi_\infty\|_\infty^2, \quad (3.27)$$

with C_Ω a constant dependent on the domain Ω . Finally, collecting all the estimates above and recalling (3.4) and (3.5), we obtain for $(\bar{y}, \bar{z}) \in S_T$ that

$$\begin{aligned} \int_0^T \|G(\bar{y}(t))\|_H^2 dt &= \sum_{j=1}^7 \int_0^T \|I_j(\bar{y})\|_H^2 dt = C \int_0^T 2 \|A^{3/2} \bar{y}\|_H^2 \|A^{1/2} \bar{y}\|_H^4 dt \\ &+ C \int_0^T (\|\nabla \varphi_\infty\|_\infty^2 + \|\Delta \varphi_\infty\|_\infty^2 + 2 \|\varphi_\infty\|_\infty^2) \|A^{3/2} \bar{y}\|_H^2 \|A^{1/2} \bar{y}\|_H^2 dt \\ &+ C \|g\|_{2,\infty}^2 \int_0^T \|A^{3/2} \bar{y}\|_H^2 dt. \end{aligned}$$

In view of (3.9), we conclude that

$$\int_0^T \|G(\bar{y}(t))\|_H^2 dt \leq C (r^6 + \|\varphi_\infty\|_{2,\infty}^2 r^4 + \|g\|_{2,\infty}^2 r^2), \quad (3.28)$$

where we stress that C is a constant independent of φ_∞ . Going back to (3.13), we can write it in the form

$$\begin{aligned} & \|y(t)\|_{D(A^{1/2})}^2 + \|z(t)\|_{D(A^{1/4})}^2 + \int_0^t \left(\|A^{3/2}y(s)\|_H^2 + \|A^{3/4}z(s)\|_H^2 \right) ds \\ & \leq C_P \left(\|y_0\|_{D(A^{1/2})}^2 + \|z_0\|_{D(A^{1/4})}^2 + \int_0^t \|G(\bar{y}(s))\|_H^2 ds \right) \end{aligned} \quad (3.29)$$

where C_P is a constant depending on ν, γ, l and $\|\varphi_\infty\|_{L^2(\Omega)}$, and we would like to impose that the left-hand side is $\leq r^2$. To this aim it suffices that the right-hand side is $\leq r^2$. On account of (3.28) we see that the latter condition holds provided that

$$\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})} \leq \rho$$

and

$$C_P \rho^2 + C C_P (r^6 + \|\varphi_\infty\|_{2,\infty}^2 r^4 + \|g\|_{2,\infty}^2 r^2) \leq r^2,$$

where C_P and C are the precise constants entering in (3.29) and (3.28). We notice that the first condition coincides with (3.6). Then, if we assume that

$$C_P \rho^2 \leq \frac{1}{2} r^2 \quad \text{that is, } \rho \leq (2C_P)^{-1/2} r \quad (3.30)$$

a sufficient condition for our bound is

$$r^4 + \|\varphi_\infty\|_{2,\infty}^2 r^2 + \|g\|_{2,\infty}^2 - \frac{1}{2CC_P} \leq 0.$$

This is satisfied provided that

$$\|g\|_{2,\infty}^2 \leq (2C_P C)^{-1/2} \quad \text{and } r \leq r_1 \quad (3.31)$$

where $r_1 > 0$ is given by

$$r_1^2 = \frac{-\|\varphi_\infty\|_{2,\infty}^2 + \sqrt{D_\infty}}{2} \quad (3.32)$$

with

$$D_\infty = \|\varphi_\infty\|_{2,\infty}^4 - 4(\|g\|_{2,\infty}^2 - (2C_P C)^{-1}).$$

Notice that the first condition in (3.31) implies that D_∞ is nonnegative and that r_1 is well-defined. Both D_∞ and r_1 depend on the full norm $\|\varphi_\infty\|_{2,\infty}$, i.e., on $\|g_\infty\|_\infty$ and χ_∞ , but they are independent of T .

Now, we look for a sufficient condition for it. By (3.26)-(3.27) and (3.5) we have

$$\|g\|_{2,\infty} \leq C_\Omega \bar{g}_\infty \leq C_\Omega (\|\varphi_\infty\|_\infty \chi_\infty + \chi_\infty^2).$$

Hence, the first inequality in (3.31) holds if $C_\Omega (\|\varphi_\infty\|_\infty \chi_\infty + \chi_\infty^2) \leq (2C_P C)^{-1/2}$, that is

$$\chi_\infty^2 + \|\varphi_\infty\|_\infty \chi_\infty - C_\Omega^{-1} (2C_P C)^{-1/2} \leq 0.$$

But this is true whenever

$$\chi_\infty \leq \chi'_0 := \frac{-\|\varphi_\infty\|_\infty + \sqrt{\|\varphi_\infty\|_\infty^2 + 4C_\Omega^{-1}(2C_P C)^{-1/2}}}{2}. \quad (3.33)$$

We stress that χ'_0 depends on ν, l, γ, Ω and $\|\varphi_\infty\|_\infty$, but it is independent of T .

So, if we assume (3.5)-(3.6) with χ'_0 given by (3.33) and ρ and r satisfying (3.30) and the second constraint in (3.31), coming back to (3.29) we obtain

$$\|y(t)\|_{D(A^{1/2})}^2 + \|z(t)\|_{D(A^{1/4})}^2 + \int_0^t \left(\|A^{3/2}y(s)\|_H^2 + \|A^{3/4}z(s)\|_H^2 \right) ds \leq r^2 \quad (3.34)$$

for all $t \in [0, T]$.

In conclusion, if we fix r and ρ satisfying (3.31) and (3.30), and if $\|\nabla\varphi_\infty\|_\infty + \|\Delta\varphi_\infty\|_\infty$ is small as specified above we have that $(y, z) \in S_T$, hence Ψ_T maps S_T into S_T . It is understood that r and ρ are fixed in the sequel (and do not depend on T).

ii) Let $(y, z) = \Psi_T(\bar{y}, \bar{z})$, with $(\bar{y}, \bar{z}) \in S_T$. We observe that (y, z) and $\frac{d}{dt}(y, z)$ remain bounded in $L^2(0, T; D(A^{3/2}) \times D(A^{3/4}))$ and $W^{1,2}([0, T]; (D(A) \times D(A^{1/2}))')$, respectively. Here we used Proposition 2.1 (replacing the term $1_\omega^* U(t)$ in (2.14) by $\mathcal{G}(\bar{y}(t))$ in (3.10)) and the estimate (2.22) in which the last term on the right-hand side, $\int_0^T \|1_\omega^* U(s)\|_{\mathcal{H}}^2 ds$, is replaced by the integral $\int_0^T \|G(\bar{y}(t))\|_H^2 dt$ in (3.28). Since $D(A^{3/2}) \times D(A^{3/4})$ is compactly embedded in $D(A^{3/2-\varepsilon}) \times H$ it follows by Lions-Aubin lemma (see [24], p. 58) that the set $\Psi_T(S_T)$ is relatively compact in $L^2(0, T; D(A^{3/2-\varepsilon}) \times H)$.

iii) Let $(\bar{y}_n, \bar{z}_n) \in S_T$, $(\bar{y}_n, \bar{z}_n) \rightarrow (\bar{y}, \bar{z})$ strongly in $L^2(0, T; D(A^{3/2-\varepsilon}) \times H)$, as $n \rightarrow \infty$. We have to prove that

$$\Psi_T(\bar{y}_n, \bar{z}_n) \rightarrow \Psi_T(\bar{y}, \bar{z}) \text{ strongly in } L^2(0, T; D(A^{3/2-\varepsilon}) \times H).$$

The solution (y_n, z_n) to (3.10) corresponding to (\bar{y}_n, \bar{z}_n) belongs to the spaces specified in (3.12) and satisfies the estimate (3.34). Also, $\{\frac{d}{dt}(y_n, z_n)\}_n$ is bounded in $L^2(0, T; (D(A) \times D(A^{1/2}))')$. Hence, on a subsequence $\{n \rightarrow \infty\}$ we have

$$y_n \rightarrow y \text{ weakly in } L^2(0, T; D(A^{3/2})) \text{ and weak* in } L^\infty(0, T; D(A^{1/2})),$$

$$z_n \rightarrow z \text{ weakly in } L^2(0, T; D(A^{3/4})) \text{ and weak* in } L^\infty(0, T; D(A^{1/4})),$$

$$\frac{dy_n}{dt} \rightarrow \frac{dy}{dt} \text{ weakly in } L^2(0, T; (D(A))'),$$

$$\frac{dz_n}{dt} \rightarrow \frac{dz}{dt} \text{ weakly in } L^2(0, T; (D(A^{1/2}))').$$

Then, by Aubin-Lions lemma

$$(y_n, z_n) \rightarrow (y, z) \text{ strongly in } L^2(0, T; D(A^{3/2-\varepsilon}) \times H).$$

Let us to show that

$$G(\bar{y}_n) \rightarrow G(\bar{y}) \text{ weakly in } L^2(0, T; H),$$

by treating the terms in (3.4). Taking into account that $D(A^{3/2-\varepsilon}) \subset H^{3-2\varepsilon}(\Omega)$ and that $\overline{y_n} \rightarrow \overline{y}$ strongly in $L^2(0, T; D(A^{3/2-\varepsilon}))$ we infer that

$$\overline{y_n} \rightarrow \overline{y}, \quad \Delta \overline{y_n} \rightarrow \Delta \overline{y} \text{ strongly in } L^2(0, T; L^2(\Omega)),$$

$$\nabla \overline{y_n} \rightarrow \nabla \overline{y} \text{ strongly in } (L^2(0, T; L^2(\Omega)))^d$$

and so, on a subsequence they tend a.e. in Q . This implies that

$$I_j(\overline{y_n}) \rightarrow I_j(\overline{y}), \text{ a.e. on } Q, \text{ for } j = 1, \dots, 7.$$

On the other hand, by estimates (3.15)-(3.22) we conclude that $\{I_j(\overline{y_n})\}_n$ is bounded in $L^2(0, T; H)$, and selecting a subsequence we have

$$I_j(\overline{y_n}) \rightarrow \zeta_j \text{ weakly in } L^2(0, T; H), \text{ for } j = 1, \dots, 7.$$

Thus, we deduce that $\zeta_j = I_j(\overline{y})$, a.e. on Q , for $j = 1, \dots, 7$.

Moreover, writing the weak form of (3.10) corresponding to $(\overline{y_n}, \overline{z_n})$ and passing to the limit we get that $(y, z) = \Psi_T(\overline{y}, \overline{z})$. As the same holds for any subsequence this ends the proof of the continuity of Ψ_T .

Then, by the Schauder fixed point theorem, applied to the mapping Ψ_T on the space $L^2(0, T; D(A^{3/2-\varepsilon}) \times H)$, it follows that problem (3.10) has at least a solution on the interval $[0, T]$, $(y, z) \in S_T$.

Step 2. We prove here the uniqueness of the solution on $[0, T]$. Let us consider the nonlinear system (1.28)-(1.31) written in terms of φ and σ , where by (2.77)

$$(1_\omega^* v(t), 1_\omega^* u(t)) = -BB^* R(y(t), z(t)), \quad t \in (0, T). \quad (3.35)$$

For the uniqueness proof we need to prove that

$$BB^* \text{ is linear continuous from } V' \times V' \rightarrow V' \times V', \quad (3.36)$$

while we already know that

$$R \text{ is linear continuous from } D(A^{1/2}) \times D(A^{1/4}) \rightarrow (D(A^{1/2}) \times D(A^{1/4}))', \quad (3.37)$$

the latter being known by the definition of R (according to (2.51)).

In what concerns (3.36), using the definitions of B and B^* , we have that if $q = (q_1, q_2) \in V' \times V'$, then

$$BB^* q = \begin{bmatrix} \sum_{i=1}^N 1_\omega^* \varphi_i (\langle q_1, 1_\omega^* \varphi_i \rangle + \langle q_2, 1_\omega^* \psi_i \rangle) \\ \sum_{i=1}^N 1_\omega^* \psi_i (\langle q_1, 1_\omega^* \varphi_i \rangle + \langle q_2, 1_\omega^* \psi_i \rangle) \end{bmatrix} \quad (3.38)$$

is well defined because 1_ω^* is a multiplier in V . Then, it is easily seen that

$$\|BB^* q\|_{V' \times V'} \leq C \|q\|_{V' \times V'} \text{ for } q \in V' \times V'. \quad (3.39)$$

Now, we rewrite (1.28)-(1.29) in terms of the operator A and have

$$\varphi_t + \nu A^2 \varphi + A(\varphi^3 + (l-1-2\nu)\varphi - \gamma\sigma) - \varphi^3 - (l-1-\nu)\varphi + \gamma\sigma = 1_\omega^* v, \quad (3.40)$$

$$\sigma_t + A(\sigma - \gamma\varphi) - \sigma + \gamma\varphi = 1_\omega^* u, \quad (3.41)$$

with the boundary and initial conditions (1.30)-(1.31).

Assume that there are two solutions (φ^i, σ^i) , $i = 1, 2$, corresponding to $U_i = (v_i, u_i)$ with $1_\omega^* U_i = -BB^* R(y_i, z_i)$, $i = 1, 2$. We take the difference of the equations (3.40) and test it by $A^{-1}(\varphi^1 - \varphi^2)$. Then, test the difference of equations (3.41) by $\lambda(\sigma^1 - \sigma^2)$, where $\lambda > 0$ is a coefficient to be chosen later. We use the simplified notation $\varphi = \varphi^1 - \varphi^2$, $\sigma = \sigma^1 - \sigma^2$, $v = v_1 - v_2$, $u = u_1 - u_2$.

Moreover, we see the operator A also from V to V' defined by

$$\langle Aw, \psi \rangle_{V', V} = \int_{\Omega} (\nabla w \cdot \nabla \psi + w\psi) dx \quad \text{for } \psi \in V.$$

For the first computation we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varphi(t)\|_{V'}^2 + \nu \|\varphi(t)\|_V^2 + \int_{\Omega} ((\varphi^1)^3 - (\varphi^2)^3) (\varphi^1 - \varphi^2) dx \\ & \leq (2\nu + 1 - l) \int_{\Omega} \varphi^2 dx + \gamma \int_{\Omega} \sigma(\varphi - A^{-1}\varphi) dx \\ & \quad + \int_{\Omega} ((\varphi^1)^3 - (\varphi^2)^3) A^{-1}(\varphi^1 - \varphi^2) dx + \int_{\Omega} (l-1-\nu)\varphi A^{-1}\varphi dx \\ & \quad + \langle 1_\omega^* v(t), A^{-1}\varphi(t) \rangle_{V', V}, \end{aligned} \quad (3.42)$$

where we have used the property that $\langle A\varphi, \varphi \rangle_{V', V} = \|\varphi\|_V^2$ and $\langle w, A^{-1}w \rangle_{V', V} = \|w\|_{V'}^2$ for $w \in V'$.

Let us estimate each term on the right-hand side:

$$(2\nu + 1 - l) \int_{\Omega} \varphi^2 dx \leq |2\nu + 1 - l| \|\varphi(t)\|_{V'} \|\varphi(t)\|_V \leq \frac{\nu}{8} \|\varphi(t)\|_V^2 + C \|\varphi(t)\|_{V'}^2.$$

By the Young inequality

$$\begin{aligned} \gamma \int_{\Omega} \sigma(\varphi - A^{-1}\varphi) dx & \leq \gamma \|\sigma(t)\|_{V'} (\|\varphi(t)\|_V + \|A^{-1}\varphi(t)\|_V) \\ & \leq \frac{\nu}{8} \|\varphi(t)\|_V^2 + C(\|\sigma(t)\|_H^2 + \|\varphi(t)\|_{V'}^2), \end{aligned}$$

where we have used the continuous embedding $H \subset V'$ and the property $\|A^{-1}\varphi\|_V \leq C \|\varphi\|_{V'}$. Next, by using the assumptions $\varphi^i \in L^\infty(0, T; V)$ we have

$$\begin{aligned} & \int_{\Omega} ((\varphi^1)^3 - (\varphi^2)^3) A^{-1}(\varphi^1 - \varphi^2) dx = \int_{\Omega} \varphi ((\varphi^1)^2 + \varphi^1 \varphi^2 + (\varphi^2)^2) A^{-1}\varphi dx \\ & \leq C \|\varphi(t)\|_{L^4(\Omega)} (\|(\varphi^1)^2(t)\|_{L^2(\Omega)} + \|(\varphi^2)^2(t)\|_{L^2(\Omega)}) \|A^{-1}\varphi(t)\|_{L^4(\Omega)} \\ & \leq C \|\varphi(t)\|_V (\|\varphi^1(t)\|_{L^4(\Omega)}^2 + \|\varphi^2(t)\|_{L^4(\Omega)}^2) \|A^{-1}\varphi(t)\|_V \\ & \leq \frac{\nu}{8} \|\varphi(t)\|_V^2 + C(\|\varphi^1(t)\|_V^4 + \|\varphi^2(t)\|_V^4) \|\varphi(t)\|_{V'}^2 \\ & \leq \frac{\nu}{8} \|\varphi(t)\|_V^2 + C \|\varphi(t)\|_{V'}^2, \end{aligned}$$

by Hölder's inequality and the continuous embedding $V \subset L^4(\Omega)$. The last constant C also accounts for the norms of φ^i in $L^\infty(0, T; V)$. Furthermore,

$$\int_{\Omega} (l - 1 - \nu) \varphi A^{-1} \varphi dx \leq |l - 1 - \nu| \|\varphi(t)\|_{V'} \|A^{-1} \varphi(t)\|_V \leq C \|\varphi(t)\|_{V'}^2,$$

and finally

$$\begin{aligned} & \langle 1_{\omega}^* v(t), A^{-1} \varphi(t) \rangle_{V', V} \leq C \|v(t)\|_{V'} \|A^{-1} \varphi(t)\|_V \\ & \leq C (\|\varphi(t)\|_{D(A^{1/2})} + \|\sigma(t)\|_{D(A^{1/4})}) \|\varphi(t)\|_{V'} \\ & \leq C (\|\varphi(t)\|_V + \|\sigma(t)\|_V) \|\varphi(t)\|_{V'} \leq \frac{\nu}{8} \|\varphi(t)\|_V^2 + \frac{\lambda}{8} \|\sigma(t)\|_V^2 + C_{\lambda} \|\varphi(t)\|_{V'}^2, \end{aligned}$$

with the help of (3.35)-(3.37). Note that in the last bound we have already used the parameter $\lambda > 0$. Now, we detail the second computation which makes use of the parameter λ . We have

$$\begin{aligned} \frac{\lambda}{2} \frac{d}{dt} \|\sigma(t)\|_H^2 + \lambda \|\sigma(t)\|_V^2 & \leq \gamma \lambda \langle A \varphi(t), \sigma(t) \rangle_{V', V} + \lambda \|\sigma(t)\|_H^2 \\ & \quad + \gamma \lambda \int_{\Omega} \varphi \sigma dx + \lambda \langle 1_{\omega}^* u(t), \sigma(t) \rangle_{V', V} \end{aligned} \quad (3.43)$$

and treat the terms on the right-hand side. Thus, we see that

$$\gamma \lambda \langle A \varphi(t), \sigma(t) \rangle_{V', V} \leq \gamma \lambda \|\varphi(t)\|_V \|\sigma(t)\|_V \leq \frac{\lambda}{8} \|\sigma(t)\|_V^2 + 2\gamma^2 \lambda \|\varphi(t)\|_V^2,$$

then

$$\gamma \lambda \int_{\Omega} \varphi \sigma dx \leq \gamma \lambda \|\varphi(t)\|_{V'} \|\sigma(t)\|_V \leq \frac{\lambda}{8} \|\sigma(t)\|_V^2 + C_{\lambda} \|\varphi(t)\|_{V'}^2,$$

and using (1.42) and $V = D(A^{1/2})$ we get finally

$$\begin{aligned} & \lambda \langle 1_{\omega}^* u(t), \sigma(t) \rangle_{V', V} \leq C \lambda \|u(t)\|_{V'} \|\sigma(t)\|_V \leq C_1 \lambda (\|\varphi(t)\|_{D(A^{1/2})} + \|\sigma(t)\|_{D(A^{1/4})}) \|\sigma(t)\|_V \\ & \leq C_2 \lambda \|\varphi(t)\|_V \|\sigma(t)\|_V + C_3 \lambda \|\sigma(t)\|_H^{1/2} \|\sigma(t)\|_V^{3/2} \\ & \leq \frac{\lambda}{8} \|\sigma(t)\|_V^2 + C_4 \lambda \|\varphi(t)\|_V^2 + C_{\lambda} \|\sigma(t)\|_H^2. \end{aligned}$$

Note that the Young inequality with two pairs of exponents $(2, 2)$ and $(4, \frac{4}{3})$ has been used.

Next, we sum (3.42) and (3.43) by observing that the last integral on the left-hand side of (3.42) is nonnegative and taking into account the estimates in all terms. We infer that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\varphi(t)\|_{V'}^2 + \lambda \|\sigma(t)\|_H^2) + \left(\frac{\nu}{2} - (2\gamma^2 + C_4) \lambda \right) \|\varphi(t)\|_V^2 + \frac{\lambda}{2} \|\sigma(t)\|_V^2 \\ & \leq C \|\varphi(t)\|_{V'}^2 + C \|\sigma(t)\|_H^2 + C_{\lambda} \|\varphi(t)\|_{V'}^2 + C_{\lambda} \|\sigma(t)\|_H^2, \text{ a.e. } t \in (0, T). \end{aligned}$$

At this point, we can choose

$$\lambda = \frac{\nu}{4(2\gamma^2 + C_4)},$$

then integrate from 0 to t and apply the Gronwall lemma. Thus we deduce that $\varphi = 0$, $\sigma = 0$, whence uniqueness follows.

Continuation of the existence proof on $(0, \infty)$. At the end of these two steps, relying on the existence and uniqueness of the solution on $[0, T]$, with T arbitrary, we show that (3.1) has a unique solution. We recall that r_1 is independent of T . For any $r \leq r_1$ (see (3.32)) we introduce

$$S_\infty = \left\{ (y, z) \in L^2(0, \infty; H \times H); \sup_{t \in (0, \infty)} \left(\|y(t)\|_{D(A^{1/2})}^2 + \|z(t)\|_{D(A^{1/4})}^2 \right) + \int_0^\infty \left(\|A^{3/2}y(t)\|_H^2 + \|A^{3/4}z(t)\|_H^2 \right) dt \leq r^2 \right\}. \quad (3.44)$$

By assuming (3.30) and (3.6) we show that there exists a unique solution on $[0, \infty)$ which also belongs to S_∞ .

Consider the functions $(y, z) : [0, \infty) \rightarrow D(A^{3/2-\varepsilon}) \times H$, defined by

$$(y(t), z(t)) = (y_T(t), z_T(t)), \text{ for any } t \in [0, T],$$

where $(y_T(t), z_T(t))$ denotes here the solution on $[0, T]$ constructed at Steps 1 and 2. By the uniqueness proof, $(y_T(t), z_T(t)) = (y_{T'}(t), z_{T'}(t))$ on $[0, T] \subset [0, T']$ and so (y, z) is well defined. Moreover, by the first part of the proof, under the assumption (3.6) it follows that $(y, z) \in S_\infty$ and it is the solution to (3.1) satisfying (3.7).

Step 3. To prove the stabilization result we multiply equation (3.1) by $R(y(t), z(t))$ scalarly in $H \times H$. Since R is symmetric as an unbounded operator in $H \times H$ and the Riccati equation (2.59) and the estimate (2.58) hold, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (R(y(t), z(t)), (y(t), z(t)))_{H \times H} \\ & + \frac{1}{2} \left(\|A^{3/2}y(t)\|_H^2 + \|A^{3/4}z(t)\|_H^2 + \|B^*R(y(t), z(t))\|_{\mathbb{R}^N}^2 \right) \\ & \leq \|G(y(t))\|_{H \times H} \|R(y(t), z(t))\|_{H \times H} \leq \|G(y(t))\|_H \|R(y(t), z(t))\|_{H \times H} \end{aligned} \quad (3.45)$$

a.e. $t \in (0, T)$. This relation is used to compute an estimate for the norm $\|(y(t), z(t))\|_{H \times H}$ using (2.58), rewritten for convenience in the following way

$$\|R(y(t), z(t))\|_{H \times H} \leq C_R \|(y(t), z(t))\|_{D(A) \times D(A^{1/2})} \leq C_R (\|Ay(t)\|_H + \|A^{1/2}z(t)\|_H). \quad (3.46)$$

Now, we estimate the right-hand side RHS of (3.45) and for simplicity, we shall write (y, z) instead of $(y(t), z(t))$. By recalling (3.4) we see that

$$RHS \leq C_R \sum_{j=1}^7 \|I_j(y(t))\|_H (\|Ay(t)\|_H + \|A^{1/2}z(t)\|_H) \quad (3.47)$$

and we treat each term RHS_j of the sum, separately, as done in (3.14)-(3.22).

We have

$$\begin{aligned} RHS_1 &= \|I_1\|_H \|R(y, z)\|_{H \times H} \leq C \|A^{3/2}y\|_H^{(2\alpha_1+\alpha_2-1)/2} \|A^{1/2}y\|_H^{(7-(2\alpha_1+\alpha_2))/2} \|Ay\|_H \\ &+ C \|A^{3/2}y\|_H^{(2\alpha_1+\alpha_2-1)/2} \|A^{1/2}y\|_H^{(7-(2\alpha_1+\alpha_2))/2} \|A^{1/2}z\|_H \\ &\leq C \|A^{3/2}y\|_H^{(2\alpha_1+\alpha_2+1)/2} \|A^{1/2}y\|_H^{(7-(2\alpha_1+\alpha_2))/2} \\ &+ C \|A^{3/2}y\|_H^{(2\alpha_1+\alpha_2-1)/2} \|A^{1/2}y\|_H^{(7-(2\alpha_1+\alpha_2))/4} \|A^{1/2}y\|_H^{(7-(2\alpha_1+\alpha_2))/4} \|A^{1/2}z\|_H. \end{aligned}$$

We used the relation (1.43). We should take into account that the calculations before are true for $\alpha_1 \geq \frac{9}{8}$, $\alpha_2 \geq \frac{3}{4}$ (see (3.14)). By choosing $2\alpha_1 + \alpha_2 = 3$ (e.g., for $\alpha_1 = \frac{9}{8}$ and $\alpha_2 = \frac{3}{4}$) we obtain

$$\begin{aligned} RHS_1 &\leq C \left(\|A^{3/2}y\|_H^2 \|A^{1/2}y\|_H^2 + \|A^{3/2}y\|_H \|A^{1/2}y\|_H \|A^{1/2}y\|_H \|A^{1/2}z\|_H \right) \\ &\leq C \left(\|A^{3/2}y\|_H^2 \|A^{1/2}y\|_H^2 + \|A^{3/2}y\|_H^2 \|A^{1/2}y\|_H^2 + \|A^{1/2}y\|_H^2 \|A^{1/2}z\|_H^2 \right) \\ &\leq C \left(\|A^{3/2}y\|_H^2 + \|A^{1/2}z\|_H^2 \right) \|A^{1/2}y\|_H^2. \end{aligned} \quad (3.48)$$

For RHS_2 we have

$$\begin{aligned} RHS_2 &= \|I_2\|_H \|R(y, z)\|_{H \times H} \\ &\leq C \|A^{3/2}y\|_H^{(\alpha_1+2\alpha_2-1)/2} \|A^{1/2}y\|_H^{(7-(\alpha_1+2\alpha_2))/2} (\|Ay\|_H + \|A^{1/2}z\|_H) \\ &\leq C \|A^{3/2}y\|_H^{(\alpha_1+2\alpha_2+1)/2} \|A^{1/2}y\|_H^{(7-(\alpha_1+2\alpha_2))/2} \\ &\quad + C \|A^{3/2}y\|_H^{(\alpha_1+2\alpha_2-1)/2} \|A^{1/2}y\|_H^{(7-(\alpha_1+2\alpha_2))/4} \|A^{1/2}y\|_H^{(7-(\alpha_1+2\alpha_2))/4} \|A^{1/2}z\|_H \\ &\leq C \left(\|A^{3/2}y\|_H^2 + \|A^{1/2}z\|_H^2 \right) \|A^{1/2}y\|_H^2 \end{aligned} \quad (3.49)$$

for $\alpha_1 + 2\alpha_2 = 3$ ($\alpha_1 = \frac{3}{4}$, $\alpha_2 = \frac{9}{8}$).

In the same way we get the other necessary estimates. We have

$$RHS_3 \leq C \|\nabla \varphi_\infty\|_\infty \left(\|A^{3/2}y\|_H^2 + \|A^{1/2}z\|_H^2 \right) \|A^{1/2}y\|_H, \quad (3.50)$$

$$RHS_4 \leq C \|\Delta \varphi_\infty\|_\infty \left(\|A^{3/2}y\|_H^2 + \|A^{1/2}z\|_H^2 \right) \|A^{1/2}y\|_H, \quad (3.51)$$

$$RHS_5 \leq C \|\varphi_\infty\|_\infty \left(\|A^{3/2}y\|_H^2 + \|A^{1/2}z\|_H^2 \right) \|A^{1/2}y\|_H, \quad (3.52)$$

$$RHS_6 \leq C \|\varphi_\infty\|_\infty \left(\|A^{3/2}y\|_H^2 + \|A^{1/2}z\|_H^2 \right) \|A^{1/2}y\|_H, \quad (3.53)$$

$$RHS_7 \leq C \|g_\infty\|_{2,\infty} \left(\|A^{3/2}y\|_H^2 + \|A^{1/2}z\|_H^2 \right). \quad (3.54)$$

Therefore, owing to (1.43), we see that (3.47) yields

$$RHS \leq \widehat{C} \left(\|A^{3/2}y\|_H^2 + \|A^{3/4}z\|_H^2 \right) \left(\|A^{1/2}y\|_H^2 + \|\varphi_\infty\|_{2,\infty} \|A^{1/2}y\|_H + \|g_\infty\|_{2,\infty} \right),$$

where we have marked \widehat{C} by using a special symbol for future convenience. We stress that \widehat{C} depends on Ω .

Coming back to (3.45), by ignoring a nonnegative term on the left-hand side and making use of the fact that $(y, z) \in S_\infty$ (see (3.44)) we conclude that for a.e. t

$$\begin{aligned} &\frac{d}{dt} (R(y(t), z(t)), (y(t), z(t)))_{H \times H} + \|A^{3/2}y\|_H^2 + \|A^{3/4}z\|_H^2 \\ &\leq \widehat{C} \left(\|A^{3/2}y\|_H^2 + \|A^{3/4}z\|_H^2 \right) \left(r^2 + \|\varphi_\infty\|_{2,\infty} r + \|g_\infty\|_{2,\infty} \right). \end{aligned}$$

We recall that the initial datum (y_0, z_0) , the parameter r and the target φ_∞ are already subject to some restrictions: see (3.30)-(3.33), where C_P and C are the precise constants occurring in (3.29) and (3.28). Here, we require something more. Namely, we impose that

$$C_* := 1 - \widehat{C} \left(r^2 + \|\varphi_\infty\|_{2,\infty} r + \|g_\infty\|_{2,\infty} \right) > 0. \quad (3.55)$$

If we repeat the argument we have used to obtain the restriction $\chi_\infty \leq \chi_0$ of (3.33) we see that (3.55) is satisfied whenever

$$r \leq r_2, \quad r_2 := \frac{-\|\varphi_\infty\|_{2,\infty} + \sqrt{D'_\infty}}{2}, \quad D'_\infty := \|\varphi_\infty\|_{2,\infty}^2 - 4(\|g\|_{2,\infty} - (\widehat{C})^{-1})$$

by assuming that $\|g\|_{2,\infty} \leq (\widehat{C})^{-1}$. This is true if

$$\chi_\infty \leq \chi_0'' := \frac{-\|\varphi_\infty\|_\infty + \sqrt{\|\varphi_\infty\|_\infty^2 + 4(C_\Omega \widehat{C})^{-1}}}{2} \quad (3.56)$$

with C_Ω (the same as before in Step 1) and \widehat{C} depending only on Ω . Then we set (see (3.33))

$$\chi_0 := \min \{ \chi_0', \chi_0'' \}.$$

At this point, by assuming $\chi_\infty \leq \chi_0$, we can fix $r = \min \{ r_1, r_2 \}$ and ρ satisfying (3.30).

Under the assumption $\chi_\infty \leq \chi_0$, the previous inequality implies that

$$\frac{d}{dt} (R(y(t), z(t)), (y(t), z(t)))_{H \times H} + C_* \left(\|A^{3/2} y(t)\|_H^2 + \|A^{3/4} z(t)\|_H^2 \right) \leq 0 \quad (3.57)$$

a.e. $t \in (0, \infty)$.

Next, we owe to (1.43) and (2.57) and see that

$$\|A^{3/2} y(t)\|_H^2 + \|A^{3/4} z(t)\|_H^2 \geq c_0 (R(y(t), z(t)), (y(t), z(t)))_{H \times H}$$

for some constant $c_0 > 0$ depending on the problem parameters and $\|\varphi_\infty\|_{L^2(\Omega)}$. Therefore we deduce that

$$\frac{d}{dt} (R(y(t), z(t)), (y(t), z(t)))_{H \times H} + C_* c_0 (R(y(t), z(t)), (y(t), z(t)))_{H \times H} \leq 0, \quad \text{a.e. } t \in (0, \infty), \quad (3.58)$$

and this immediately implies

$$(R(y(t), z(t)), (y(t), z(t)))_{H \times H} \leq e^{-2kt} (R(y_0, z_0), (y_0, z_0))_{H \times H} \quad (3.59)$$

where $k := \frac{C_* c_0}{2}$.

On account of (2.57) we conclude that

$$c_1 \|(y(t), z(t))\|_{D(A^{1/2}) \times D(A^{1/4})}^2 \leq c_2 e^{-2kt} \|(y_0, z_0)\|_{D(A^{1/2}) \times D(A^{1/4})}^2, \quad \text{a.e. } t > 0.$$

This is nothing but (3.8) with a precise constant in front of the exponential. Hence, we conclude the proof. \square

Proof of Theorem 1.1. The result given in Theorem 1.1 follows immediately. Assume (1.21), that is

$$\|\varphi_0 - \varphi_\infty\|_{D(A^{1/2})} + \|\theta_0 - \theta_\infty\|_{D(A^{1/4})} \leq \rho$$

and go back to (1.7)-(1.10) by transformation (1.24). Then,

$$\begin{aligned} & \|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})} = \|\varphi_0 - \varphi_\infty\|_{D(A^{1/2})} + \|\sigma_0 - \sigma_\infty\|_{D(A^{1/4})} \\ & \leq (\alpha_0 l_0 + 1) \|\varphi_0 - \varphi_\infty\|_{D(A^{1/2})} + \alpha_0 \|\theta_0 - \theta_\infty\|_{D(A^{1/4})} \leq \rho \max\{\alpha_0 l_0 + 1, \alpha_0\} =: \rho_1. \end{aligned}$$

By Theorem 3.1, for ρ_1 small enough, we get (3.7) and (3.8). The latter becomes (1.23) by using the transformation (1.24).

4 Appendix

Lemma A1 *The stationary system (1.13) has at least a solution $(\theta_\infty, \varphi_\infty)$, with θ_∞ constant and $\varphi_\infty \in H^4(\Omega) \subset C^2(\overline{\Omega})$.*

Proof. It is obvious that θ_∞ is constant and φ_∞ satisfies equation

$$\begin{aligned} \Delta(\nu \Delta \varphi_\infty - \varphi_\infty^3 + \varphi_\infty) &= 0, \\ \frac{\partial \Delta \varphi_\infty}{\partial \nu} &= \frac{\partial \varphi_\infty}{\partial \nu} = 0 \end{aligned}$$

whence

$$\nu \Delta \varphi_\infty - \varphi_\infty^3 + \varphi_\infty = C. \quad (4.1)$$

This equation has a solution $\varphi_\infty \in H^1(\Omega)$. The argument is that (4.1) is a necessary condition for φ_∞ to be a minimizer of the functional

$$\Upsilon(\varphi) = \int_{\Omega} \left(\frac{\nu}{2} |\nabla \varphi|^2 + \frac{(\varphi^2 - 1)^2}{4} + C\varphi \right) dx$$

for $\varphi \in H^1(\Omega)$. On the other hand Υ is l.s.c. and coercive on $H^1(\Omega)$, whence a minimizer exists.

It follows by (4.1) that $\Delta \varphi_\infty \in L^2(\Omega)$ and together with $\frac{\partial \varphi_\infty}{\partial \nu} = 0$ this leads to $\varphi_\infty \in H^2(\Omega)$. Then $\Delta \varphi_\infty^3 \in L^2(\Omega)$ by a simple computation and we deduce that

$$\nu \Delta \Delta \varphi_\infty = \Delta(\varphi_\infty^3 - \varphi_\infty) \in L^2(\Omega),$$

whence $\varphi_\infty \in H^4(\Omega)$ since $\nu \Delta \varphi_\infty = \varphi_\infty^3 - \varphi_\infty + C$ satisfies the homogeneous Neumann condition. Therefore, $\varphi_\infty \in C^2(\overline{\Omega})$. \square

Lemma A2 (Kalman, see [22]). *Let us consider the finite dimensional system*

$$\begin{aligned} X' + MX &= DW, \quad t \in [0, T_0] \\ X(0) &= X_0 \in \mathbb{R}^N, \end{aligned}$$

where M and D are real $N \times N$ matrices. Assume that $D^*P(t) = 0$ implies $P(t) = 0$ for all $t \in [0, T_0]$, where P is any solution to the dual backward differential system $P'(t) - M^*P(t) = 0$. Then, there exists $W \in L^2(0, T_0; \mathbb{R}^N)$ such that $X(T_0) = 0$. Moreover,

$$\int_0^{T_0} \|W(t)\|_{\mathbb{R}^N}^2 dt \leq C \|X_0\|_{\mathbb{R}^N}^2. \quad (4.2)$$

Lemma A3. Let $(E \subset F \subset E')$ be a variational triplet and let $L : E \rightarrow E'$ be a linear continuous operator such that

$$\langle Lw, w \rangle_{E', E} \geq C_1 \|w\|_E^2 - C_2 \|w\|_F^2.$$

Let $M \in \mathcal{L}(E, F)$, that is

$$\|Mw\|_F \leq C_3 \|w\|_E.$$

Then, $L + M$ is quasi m -accretive in $F \times F$.

Proof. The operator $\tilde{L} = L + M$ is continuous from E to E' and

$$\langle \tilde{L}w, w \rangle_{E', E} \geq C_1 \|w\|_E^2 - C_2 \|w\|_F^2 - C_3 \|w\|_F^2 \geq C_1 \|w\|_E^2 - C_4 \|w\|_F^2.$$

Hence, $\lambda I + \tilde{L}$ is m -accretive on $F \times F$ because

$$\left((\lambda I + \tilde{L})w, w \right)_F \geq 0, \text{ for } \lambda \geq \lambda_0$$

and equation

$$(\lambda I + \tilde{L})w = f \in F$$

has a solution for λ sufficiently large.

Lemma A4 ([19], see [28], p. 116). Let $-\mathcal{D}$ be a C_0 -semigroup generator on a Banach space \mathcal{X} and consider the Cauchy problem

$$\begin{aligned} \frac{dZ}{dt}(t) + \mathcal{D}Z(t) &= 0, \text{ a.e. } t > 0, \\ Z(0) &= Z_0. \end{aligned}$$

If

$$\int_0^\infty \|Z(t)\|_{\mathcal{X}}^2 dt \leq C \|Z_0\|_{\mathcal{X}}^2, \quad \forall Z_0 \in \mathcal{X},$$

then

$$\|Z(t)\|_{\mathcal{X}}^2 \leq Ce^{-\kappa t} \|Z_0\|_{\mathcal{X}}^2,$$

for some $\kappa > 0$.

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